

# Computing Upper Bounds for Equiangular Lines in Euclidean Spaces

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## Abstract

We develop a computable upper bound of the number of equiangular lines in various Euclidean vector spaces by combining the classical pillar decomposition with the semidefinite programming (SDP) method. The computational results show an explicit bound, which is strictly less than the well-known Gerzon's bound for general dimensions between 44 and 400. We present computable relative bounds for the angles  $1/5$  and  $1/7$ , which are considerably less than the SDP bounds (best known bounds) for a range of larger dimensions. In particular, the relative bounds for  $1/5$  are more improved by applying the known partial result.

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## 1. Introduction

In this paper, we are concerned with the maximum number of equiangular lines in Euclidean vector spaces:

**Problem Statement.** For a given positive integer  $r$  ( $r \geq 2$ ), what is the maximum number of lines in the  $r$ -dimensional Euclidean space  $\mathbb{R}^r$  such that the angle of each pair of lines equals  $\arccos(\alpha)$  for some  $\alpha > 0$ ?

By selecting a unit vector in each line in a set of equiangular lines, we can formally define it as an equiangular set of unit vectors.

**Definition 1.** We say  $X = \{x_1, \dots, x_s\} \subset \mathbb{R}^r$  is a set of equiangular lines (or, simply equiangular) if

$$\langle x_i, x_j \rangle = \begin{cases} 1 & i = j \\ \pm\alpha & i \neq j \end{cases} \quad (1)$$

for some  $\alpha > 0$ .

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By slight abuse of terminology, we will say that vectors which satisfy the equality (1) are equiangular with *angle*  $\alpha$ . The ambiguity of the sign of the inner product is due to the choice of a unit vector in each line.

We first explain why it is reasonable to ask what the maximum cardinality of an equiangular set is. It is well known that in  $\mathbb{R}^r$ , the cardinality of an equiangular set  $X$ , say  $|X|$ , is bounded above by  $\frac{r(r+1)}{2}$  and the upper bound can be attained only if  $r = 2$ ,  $r = 3$ , or  $r + 2$  is the square of an odd number (Lemmens and Seidel, 1973, Theorem 3.5). This is the so-called Gerzon's bound. It had been conjectured for many years that whenever  $r + 2$  was the square of an odd integer, that there existed a set of  $\frac{r(r+1)}{2}$  equiangular lines in  $\mathbb{R}^r$  (Section 6.1 of (Sustik et al., 2007), repeated in (Waldron, 2009)). However, counter examples have since been found. There do not exist 1128 equiangular lines in  $\mathbb{R}^{47}$  (Makhnev, 2002). Suppose  $m \equiv 2 \pmod{3}$ ,  $m$  and  $2m + 1$  are both square-free. Then for  $r = (4m + 1)^2 - 2$ , there do not exist  $\frac{r(r+1)}{2}$  equiangular lines in  $\mathbb{R}^r$  (Bannai et al., 2005). On the other hand, for general  $r$ , Gerzon's bound can not be attained. So a natural question is for a given  $r$ , what the exact maximum cardinality of an equiangular set is.

Fix an integer  $r$  ( $r \geq 2$ ) and  $\alpha > 0$ . Then  $s^\alpha(r)$  is defined to be the maximum cardinality of an equiangular set in  $\mathbb{R}^r$  with pairwise inner products  $\pm\alpha$ . Further  $s(r)$  is defined to be the maximum cardinality of any equiangular set in  $\mathbb{R}^r$ , that is,

$$s(r) \triangleq \max_{\alpha > 0} s^\alpha(r).$$

Now the problem statement can be precisely rewritten:

**Problem Statement'.** For a given positive integer  $r$  ( $r \geq 2$ ), what is  $s(r)$ ?

**Motivation.** It is a fundamental problem to study the maximum number of equiangular lines. From the point of view of applications, in certain extremal cases, equiangular sets have further desirable properties with respect to data analysis and coding theory (Conway et al., 1996; Strohmer and Heath, 2003). These special sets are called *equiangular tight frames* (ETF). An ETF is an equiangular set  $\{x_1, \dots, x_s\} \subset \mathbb{R}^r$  such that for any  $y \in \mathbb{R}^r$ ,

$$\frac{s}{r} \langle y, y \rangle = \sum_{j=1}^s \langle y, x_j \rangle^2.$$

ETFs solve a packing problem in Grassmannian space (Benedetto and Kolesar, 2006; Conway et al., 1996; Dhillon et al., 2008), are known to be optimally robust to erasures (Strohmer and Heath, 2003; Bodmann, 2007), and further have optimal coherence which is related to the appropriateness of using a set of vectors for sparse coding (Donoho and Elad, 2003; Bruckstein et al., 2009).

The theory of equiangular lines and frames is related to linear algebra (e.g., existence of certain matrices (Lemmens and Seidel, 1973; Van Lint and Seidel, 1966; Sustik et al., 2007)), group theory (e.g., difference sets (Xia et al., 2005; Ding and Feng, 2007)), geometry (e.g., regular spherical polytopes (Coxeter, 1963)), graph theory (e.g., [regular] two-graphs and strongly regular graphs (Holmes and Paulsen, 2004; Van Lint and Seidel, 1966)), combinatorics (e.g., Steiner systems (Fickus et al., 2012)) and more. Table 1 presents the currently known  $s(r)$  for the dimension  $r$  up to 43.

Some of the history of Table 1 can be found in (Lemmens and Seidel, 1973). Two interesting examples are  $r = 7$  and  $r = 23$ , which are the smallest whole numbers with the property that  $r + 2$  is the square of an odd integer. Notice that  $s(7)$  and  $s(23)$  saturate Gerzon's bound. However, it

Table 1: Maximum number of equiangular lines for small dimensions (Barg and Yu, 2014; Greaves et al., 2016; Yu, 2015; Azarija and Marc, 2016)

$r$	2	3	5	6	7–13	14
$s(r)$	3	6	10	16	28	28–29
$r$	15	16	17	18	19	20
$s(r)$	36	40–41	48–50	48–61	72–75	90–95
$r$	21	22	23–41	42	43	
$s(r)$	126	176	276	276–288	344	

is known that it can take a long time to determine  $s(r)$  for a single  $r$ . For example, it is shown  $s(43) \geq 344$  in (Taylor, 1971), while  $s(43) \leq 344$  was solved recently by semidefinite programming (SDP) (Barg and Yu, 2014). For some  $r$  ( $r = 14, 16, \dots, 20, 42$ ), we only know a range for  $s(r)$ . It turns out that shrinking these ranges is a very non-trivial task. It was proven  $s(14) < 30$  and  $s(16) < 42$  by studying Seidel matrices with exactly three distinct eigenvalues (Greaves et al., 2016). It was proven recently that  $s(19) < 76$  by studying certain classical results (Yu, 2015) and  $s(20) < 96$  by combining results from the new paper (Azarija and Marc, 2016) with (Waldron, 2009).

An attractive topic is to explore  $s(r)$  for  $r \geq 44$ . So far, for a general  $r$  ( $r \geq 44$ ), we only know  $\frac{32r^2+328r+296}{1089} \leq s(r) \leq \frac{r(r+1)}{2}$  (Greaves et al., 2016, Corollary 2.8) (although for some special cases, for instance  $r = 47$ , we know the upper bound is strictly less than Gerzon’s bound). Many known results show that Gerzon’s upper bound can be sharpened considerably. For instance,  $s(43) = 344 < \frac{43 \times 44}{2} = 946$ . One possible way to get a non-trivial upper bound is to consider  $s^\alpha(r)$  for a fixed  $\alpha$ . A nice classical result is that  $s(r)$  ( $r > 3$ ) can be solved by solving finitely many  $s^\alpha(r)$  where  $\frac{1}{\alpha}$  is an odd integer and this odd integer is bounded by  $\sqrt{2r}$  (see Theorem 2 and Corollary 1). Thus, throughout the paper, we assume  $\frac{1}{\alpha}$  is an odd integer. An upper bound for  $s^\alpha(r)$  is usually said to be a *relative bound*. A general result on relative bound is

$$s^\alpha(r) \leq \frac{r(1-\alpha^2)}{1-r\alpha^2} \quad (\text{for } r < \frac{1}{\alpha^2}). \quad (2)$$

And when  $r < \frac{1}{\alpha^2} - 2$ , the relative bound (2) is strictly less than Gerzon’s bound (Lemmens and Seidel, 1973, Theorem 3.6). Another theorem of note is Theorem 4.5 in (Lemmens and Seidel, 1973), which solves  $s^{\frac{1}{3}}(r)$  completely by decomposing an equiangular set into *pillars* (see the precise definition in Section 3.2) and studying the algebraic structure of each pillar and also the combinatorial structure when all pillars are non-empty. However, by the same spirit, the next interesting case  $\alpha = \frac{1}{5}$  is only partially solved and there is a long-standing conjecture (Lemmens and Seidel, 1973, Conjecture 5.8):

$$s^{\frac{1}{5}}(r) = \begin{cases} 276 & 23 \leq r \leq 185 \\ r + 1 + \lfloor \frac{1}{2}(r-5) \rfloor & r \geq 185 \end{cases}$$

The main progress on this conjecture has been made by graph representation and an SDP method, respectively:

- (Neumaier, 1989) There exists a sufficiently large  $N$  such that for any  $r > N$ ,

$$s^{\frac{1}{5}}(r) = r + 1 + \lfloor \frac{1}{2}(r - 5) \rfloor;$$

- (Barg and Yu, 2014) For  $23 \leq r \leq 60$ ,  $s^{\frac{1}{5}}(r) = 276$ .

The best known  $s^{\frac{1}{5}}(r)$  is summarized in (Greaves et al., 2016, Table 4). For  $\alpha \leq \frac{1}{7}$ , following the classical method, one might need to characterize the connected simple graphs with maximum eigenvalue 3 (or  $> 3$ ). The last sentence in (Neumaier, 1989) says this requires substantially stronger techniques. A good news is that the relative bounds for a general  $\alpha$  can be computed by SDP (Barg and Yu, 2014). The best known non-trivial relative bounds and upper bound of  $s(r)$  for  $44 \leq r \leq 136$  can be found in (Barg and Yu, 2014, Table 3). Notice that for  $r > 136$ , this SDP method might give a bound which is greater than Gerzon's bound.

**Contributions.** In this paper, we focus on deriving a non-trivial upper bound for a general  $r$  ( $r \geq 44$ ). Our contributions have three stages, see (C1)(C2)(C3). The contribution (C3) is our main result.

- (C1) We present a computable upper bound for  $s(r)$  ( $r > 3$ ) by combining the classical pillar decomposition (Lemmens and Seidel, 1973) with semidefinite programming (SDP) method (Barg and Yu, 2013, 2014), see Theorem 4.
- (C2) Based on Theorem 4 and a further exploration of the algebraic structure of the equiangular sets (see Lemmas 13–14), we develop a computable relative bound for  $s^{\frac{1}{7}}(r)$ , see Theorem 5. Using Theorem 4 and Lemma 13 and further applying the known partial results (see Lemma 15), we develop a computable relative bound for  $s^{\frac{1}{5}}(r)$  in Theorem 6. Figures 1–2 show these computable bounds are dramatically less than the best known relative bounds for a range of larger  $r$ .
- (C3) By applying Theorems 5–6 and solving SDP, we compute an upper bound of  $s(r)$  for  $44 \leq r \leq 400$ . The computational results show an explicit upper bound:

$$s(r) \leq \begin{cases} \frac{4r(m+1)(m+2)}{(2m+3)^2-r}, & r = 44, 45, 46, 76, 77, 78, 117, 118, 166, 222, 286, 358 \\ \frac{((2m+1)^2-2)((2m+1)^2-1)}{2}, & \text{other } r \text{ between } 44 \text{ and } 400 \end{cases}.$$

This bound is strictly less than Gerzon's bound if  $r + 2$  is not a square of an odd number, see Theorem 7. The result leads us to a conjecture on a new general upper bound, see Conjecture 1.

**Organization.** The rest of the paper is organized as follows. In Section 2, we review the SDP method introduced in (Barg and Yu, 2013) for solving the upper bounds of spherical two-distance sets. In Section 3, we introduce the pillar decomposition and prepare the lemmas. In Section 4, we prove/compute the main result (Theorem 7) and show (C1)(C2)(C3) step by step. The spirit of (Lemmens and Seidel, 1973) plays an important role in Sections 3–4. We will review the main insights in (Lemmens and Seidel, 1973) according to the flow of our proofs. In Section 5, we compare

the new computable relative bounds and the SDP bounds by experiments. We also interpret the experimental/computational details in this section. In Section 6, we conclude the paper and discuss future directions of research. Appendix A presents the new upper bound (with the relative bounds) of  $s(r)$  for  $44 \leq r \leq 400$  by Table 3. To ease comparison, Table 3 has the same style as (Barg and Yu, 2014, Table 3). Appendix B provides details for the SDP formulation.

## 2. Review spherical two-distance sets and SDP bound

In this section, we review spherical two-distance sets and the SDP method for computing the upper bound of a spherical two-distance set. A spherical two-distance set is a more general concept than an equiangular set. The SDP method is closely related to Delsarte's method (Delsarte et al., 1977; Musin, 2009) and harmonic analysis in coding theory (Bachoc and Vallentin, 2008). We provide (Yu, 2014) as a good survey for the interested readers. We are not going to repeat everything in (Yu, 2014). The main point we hope to highlight here is that an upper bound of  $s(r)$  is computable for any  $r > 3$ , see Theorem 1 and Corollary 1 below.

**Definition 2.** We say  $X = \{x_1, \dots, x_s\} \subset \mathbb{R}^r$  is a spherical two-distance set with mutual inner products  $\alpha, \beta$  if

$$\langle x_i, x_j \rangle = \begin{cases} 1 & i = j \\ \alpha \text{ or } \beta & i \neq j \end{cases}$$

for some  $\alpha, \beta \in \mathbb{R}$ .

We use the notation  $s^{\alpha, \beta}(r)$  to denote the maximum cardinality of a spherical two-distance set with mutual inner products  $\alpha, \beta$  in  $\mathbb{R}^r$ . Usually, we assume  $\alpha > \beta$ . As we see,  $s^\alpha(r)$  is equivalent to  $s^{\alpha, -\alpha}(r)$ .

**Theorem 1.** (Barg and Yu, 2013, Theorem 3.1) The upper bound of  $s^{\alpha, \beta}(r)$  is given by the solution of a semidefinite programming problem, denoted by  $\mathbf{sdp}(r, \alpha, \beta)$ .

The SDP formulation can be found in (Barg and Yu, 2013, Theorem 3.1). The history on where this formulation comes from can also be found in (Barg and Yu, 2013). For the interested readers, we provide these details in Appendix B.

**Theorem 2.** (Larman et al., 1977, Theorem 2) Suppose  $X \subset \mathbb{R}^r$  is a spherical two-distance set with mutual inner products  $\alpha, \beta$ . If  $|X| > 2r + 3$ , then there exists a positive integer  $K$  ( $2 \leq K \leq \frac{1+\sqrt{2r}}{2}$ ) such that  $\beta = \frac{K\alpha-1}{K-1}$ .

We remark that if a spherical two-distance set  $X \subset \mathbb{R}^r$  is equiangular, then  $\beta = -\alpha$  and Theorem 2 implies that if  $|X| > 2r + 3$ , then there exists a positive integer  $K$  ( $2 \leq K \leq \frac{1+\sqrt{2r}}{2}$ ) such that  $\alpha = \frac{1}{2K-1}$ . This conclusion is consistent with (Lemmens and Seidel, 1973, Theorem 3.4), which states that if  $|X| > 2r$ , then  $\frac{1}{\alpha}$  is an odd number. Furthermore, Theorem 2 gives an upper bound for the odd number  $\frac{1}{\alpha}$ . That means for any  $r$ , in order to compute the upper bound of  $s(r)$ , we only need to check the upper bound of  $s^\alpha(r)$  for finitely many  $\alpha = \frac{1}{3}, \dots, \frac{1}{2K-1}$  where  $K$  is the largest positive integer such that  $(2K-1)^2 \leq 2r$ . Note also (Lemmens and Seidel, 1973, Theorem 4.5) shows  $s^{\frac{1}{3}}(r) \leq 2r - 2 < 2r + 3$  for any  $r > 3$ . These observations from the classical results are summarized as Corollary 1 below.

**Corollary 1.** For any  $r > 3$ ,  $s(r) \leq \max \left( 2r + 3, s^{\frac{1}{5}}(r), \dots, s^{\frac{1}{2K-1}}(r) \right)$  where  $K$  is the largest positive integer such that  $(2K - 1)^2 \leq 2r$ .

**Remark 1.** If we replace the hypothesis “ $r > 3$ ” in Corollary 1 with “ $r \geq 15$ ”, then the “ $\leq$ ” in the conclusion can be replaced by “ $=$ ” (Yu, 2014, Section 3.1).

By Theorem 1 and Corollary 1, we can always compute an upper bound of  $s(r)$  by solving  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5}), \dots, \mathbf{sdp}(r, \frac{1}{2K-1}, -\frac{1}{2K-1})$ . Big progress was made by the this spirit in (Barg and Yu, 2014) in proving  $s^{\frac{1}{5}}(r) \leq 276$  ( $24 \leq r \leq 60$ ) and  $s(43) \leq 344$ . However, it is seen that  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  is greater than the Gerzon’s bound for  $r = 137$ – $139$  (Barg and Yu, 2014, Table 3). We also provide evidence in Section 5 that for  $\alpha = \frac{1}{5}, \frac{1}{7}$ , the SDP bound might not guarantee a non-trivial upper bound if  $r$  is sufficiently large. We doubt this will happen for all  $\alpha$ . In the rest of the paper, we focus on solving this problem by combining pillar decomposition with the SDP method.

### 3. Pillar decomposition

#### 3.1. Setup

Throughout the discussion below, we assume  $r > 3$  and  $\alpha \leq \frac{1}{3}$ . The notation  $s^\alpha$  and  $s^{\alpha, \beta}$  is used instead of  $s^\alpha(r)$  and  $s^{\alpha, \beta}(r)$  when  $r$  is clear. For any finite set  $S$ , we denote the cardinality of  $S$  by  $|S|$ . We first present an easy observation. As what has been pointed out at the very beginning of (Lemmens and Seidel, 1973, Section 4), we have Lemma 1.

**Lemma 1.** (Lemmens and Seidel, 1973) Suppose  $X \subset \mathbb{R}^r$  is equiangular. If there exist  $p_1, \dots, p_k \in X$  such that  $\langle p_i, p_j \rangle = -\alpha$  for any  $1 \leq i < j \leq k$ , then  $k \leq \frac{1}{\alpha} + 1$ .

We define  $E^\alpha(r)$  (or, simply  $E^\alpha$  when  $r$  is clear) as the set of all equiangular line sets  $X$  with the angle  $\alpha$  in  $\mathbb{R}^r$ . According to the definition of  $E^\alpha$ , we have  $s^\alpha = \max_{X \in E^\alpha} |X|$ . For any positive integer  $k$ , we define two subsets of  $E^\alpha$ , say  $E_{\geq k}^\alpha$  and  $E_{=k}^\alpha$ :

$$E_{\geq k}^\alpha \triangleq \{X \in E^\alpha \mid \exists p_1, \dots, p_k \in X \text{ such that for any } 1 \leq i < j \leq k, \langle p_i, p_j \rangle = -\alpha\}$$

$$E_{=k}^\alpha \triangleq \{X \in E_{\geq k}^\alpha \mid \forall p_1, \dots, p_k, p_{k+1} \in X, \exists i, j, \text{ such that } \langle p_i, p_j \rangle = \alpha\}$$

Let us denote  $\max_{X \in E_{\geq k}^\alpha} |X|$  and  $\max_{X \in E_{=k}^\alpha} |X|$  by  $s_{\geq k}^\alpha$  and  $s_{=k}^\alpha$  respectively. Lemma 2 below says that in order to compute  $s^\alpha$ , we only need to compute  $s_{=k}^\alpha$  for each  $k = 1, \dots, \frac{1}{\alpha} + 1$ . Lemma 3 gives the upper bound for  $s_{=1}^\alpha$  by the structure of the so-called Gramian matrix.

**Lemma 2.**  $s^\alpha = \max \left( s_{=1}^\alpha, s_{=2}^\alpha, \dots, s_{=\frac{1}{\alpha}+1}^\alpha \right)$

*Proof.* From Lemma 1, we see for all  $k > \frac{1}{\alpha} + 1$ ,  $E_{=k}^\alpha = E_{\geq k}^\alpha = \emptyset$ . So

$$E^\alpha = \bigcup_{k=1}^{\frac{1}{\alpha}+1} E_{=k}^\alpha, \text{ and } E_{=i}^\alpha \cap E_{=j}^\alpha = \emptyset \text{ (} i \neq j \text{)} \quad (3)$$

By the formula (3), we directly have the conclusion.  $\square$

**Definition 3.** For any  $X = \{x_1, \dots, x_s\} \subset \mathbb{R}^r$ , the Gramian matrix of  $X$  is the matrix of mutual inner products of the vectors in  $X$ , namely

$$\begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle & \dots & \langle x_1, x_s \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \dots & \langle x_2, x_s \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_s, x_1 \rangle & \langle x_s, x_2 \rangle & \dots & \langle x_s, x_s \rangle \end{pmatrix},$$

denoted by  $G(x_1, \dots, x_s)$  or  $G(X)$ .

For any finite set  $X \subset \mathbb{R}^r$ ,  $G(X)$  is naturally symmetric and positive semidefinite. If  $X$  is equiangular with angle  $\alpha$ , then  $G$  has 1's along the diagonal and  $\pm\alpha$  as the off-diagonal entries. Following Seidel's spirit, we will often decompose matrices using building blocks of the  $s \times s$  identity matrix  $I_s$  and the  $s \times s$  all-one matrix  $J_s$ , or simply  $I$  (or, respectively  $J$  when  $s$  is clear from context).

**Lemma 3.**  $s_{=1}^\alpha \leq r$ .

*Proof.* By the definition of  $E_{=1}^\alpha$ , it is seen that for any  $X \in E_{=1}^\alpha$ , all off-diagonal entries of  $G(X)$  are  $\alpha$  and hence  $G(X) = (1 - \alpha)I + \alpha J$  is full rank. Therefore,  $|X| \leq r$  and hence  $s_{=1}^\alpha \leq r$ .  $\square$

In the next two subsections, we will derive the upper bound of  $s_{=k}^\alpha$  for  $2 \leq k \leq \frac{1}{\alpha} + 1$  by studying the structures. We define some notations for the discussions.

- For any  $P \subset \mathbb{R}^r$ , let  $\Pi_P$  denote the linear subspace of  $\mathbb{R}^r$  spanned by  $P$  and let  $\Pi_P^\perp$  denote the orthogonal complement of  $\Pi_P$  in  $\mathbb{R}^r$ .
- For any integer  $k$  ( $2 \leq k \leq \frac{1}{\alpha} + 1$ ) and for any integer  $i$  ( $0 \leq i < \frac{k}{2}$ ), define  $\xi_{k,i}$  as the set of  $(1, -1)$ -vectors in  $\mathbb{R}^k$  with  $i$  1-coordinates and  $k - i$   $(-1)$ -coordinates. In addition, for any even  $k$ , define  $\xi_{k, \frac{k}{2}}$  as the set of  $(1, -1)$ -vectors  $\{(1, \epsilon) | \epsilon \in \xi_{k-1, \frac{k}{2}-1}\}$ . Denote  $|\xi_{k,i}|$  by  $b_{k,i}$ . Note

$$b_{k,i} \triangleq \begin{cases} \binom{k}{i} & 0 \leq i < \frac{k}{2} \\ \binom{k}{\frac{k}{2}}/2 & k \text{ is even and } i = \frac{k}{2} \end{cases}$$

Assume  $\xi_{k,i} = \{\epsilon_{k,i,1}, \dots, \epsilon_{k,i,b_{k,i}}\}$ .

**Example 1.**  $\xi_{3,1} = \{ (1, -1, -1), (-1, 1, -1), (-1, -1, 1) \}.$   
 $\xi_{4,2} = \{ (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1) \}.$

Note when we say “a vector” is in  $\mathbb{R}^r$ , we assume the vector is a row vector. We apply the transpose symbol  $^T$  to get a column vector. For instance,  $\epsilon_{4,2,1} = (1, 1, -1, -1)$  in Example 1 is a row vector and  $\epsilon_{4,2,1}^T$  is a column vector.

### 3.2. $k = \frac{1}{\alpha} + 1$

When  $k = \frac{1}{\alpha} + 1$ , the structure of  $s_{=k}^\alpha$  was studied in detail in (Lemmens and Seidel, 1973). In this subsection, we revisit these important results using our notation. By the definition of  $E_{=k}^\alpha$ , for any  $X \in E_{=k}^\alpha$ , there exists  $P = \{p_1, \dots, p_k\} \subset X$  such that for any  $i \neq j$ ,  $\langle p_i, p_j \rangle = -\alpha$ . In this case ( $k = \frac{1}{\alpha} + 1$ ),  $p_1, \dots, p_k$  form a  $(k - 1)$ -simplex, and  $\sum_{i=1}^k p_i = 0$ . For any  $h \in \Pi_P$ , the

linear manifold  $h + \Pi_P^\perp$  is the so-called *pillar* of  $h$  in (Lemmens and Seidel, 1973). The main idea of (Lemmens and Seidel, 1973) is to consider an  $X \in E_{=k}^\alpha$ , which can be decomposed as a partition:

$$X = P \bigcup_{j=1}^{b_{k, \frac{k}{2}}} P_{k, \frac{k}{2}, j} \quad (4)$$

where

$$P_{k, \frac{k}{2}, j} \triangleq \{x \in X | (\langle x, p_1 \rangle, \dots, \langle x, p_k \rangle) = \alpha \epsilon_{k, \frac{k}{2}, j}\}, \text{ for } j = 1, \dots, b_{k, \frac{k}{2}}.$$

The definition of  $P_{k, \frac{k}{2}, j}$  depends on  $\alpha$ . So the full notation should be  $P_{k, \frac{k}{2}, j}(\alpha)$ , but we use  $P_{k, \frac{k}{2}, j}$  instead when  $\alpha$  is clear. Note that each  $P_{k, \frac{k}{2}, j}$  is a subset of a pillar  $h_{k, \frac{k}{2}, j} + \Pi_P^\perp$  where

$$h_{k, \frac{k}{2}, j} \triangleq \frac{\alpha}{1 + \alpha} \left( \epsilon_{k, \frac{k}{2}, j}^{(1)} p_1 + \dots + \epsilon_{k, \frac{k}{2}, j}^{(k)} p_k \right), \text{ where again } \epsilon_{k, \frac{k}{2}, j}^{(i)} \text{ is the } i\text{-th coordinate of } \epsilon_{k, \frac{k}{2}, j}.$$

That mean for any  $x \in P_{k, \frac{k}{2}, j}$ ,  $x$  can be written as  $x = h_{k, \frac{k}{2}, j} + c$  for some  $c \in \Pi_P^\perp$ . Note  $h_{k, \frac{k}{2}, j} \in P$ . So for any two  $x_1, x_2 \in P_{k, \frac{k}{2}, j}$ , their projections onto  $P$  are the same. By Perron-Frobenius theory, it is proven that  $|P_{k, \frac{k}{2}, j}| \leq r - k + 1 + \lfloor 2\alpha \frac{r-k+1}{1-\alpha} \rfloor$  (Lemmens and Seidel, 1973, Theorem 4.1). As a result of (4), for a general  $\alpha$ , Theorem 3 below is hidden in (Lemmens and Seidel, 1973).

**Theorem 3.**  $s_{=k}^\alpha \leq k + \frac{1}{2} \binom{k}{\frac{k}{2}} \left( r - k + 1 + \lfloor 2\alpha \frac{r-k+1}{1-\alpha} \rfloor \right)$  where  $k = \frac{1}{\alpha} + 1$ .

**Remark 2.** Not every  $X \in E_{=k}^\alpha$  can be decomposed as (4). But by considering so-called switching equivalence, it suffices to study  $X$  with decomposition (4). We will explain this precisely by Lemma 5 in Section 3.3.

For  $\alpha = \frac{1}{3}, \frac{1}{5}$ , the upper bound in Theorem 3 can be reduced significantly by applying spectral graph theory. For instance,  $s_{=4}^{\frac{1}{3}} \leq 2(r-1)$  (Lemmens and Seidel, 1973, Theorem 4.5). What is deeply hidden in its proof is that the only connected simple graph with maximum eigenvalue 1 is  $K_2$  (the complete simple graph with 2 vertices). Lemma 4 below is proposed by the fact that the connected simple graphs with maximum eigenvalue 2 only have 5 patters (Lemmens and Seidel, 1973, Theorem 5.1).

**Lemma 4.** (Lemmens and Seidel, 1973, Theorem 5.7)  $s_{=6}^{\frac{1}{5}} \leq \max(276, r + 1 + \lfloor \frac{1}{2}(r-5) \rfloor)$ .

3.3.  $k = 2, 3, \dots, \frac{1}{\alpha}$

In this subsection, we study the structure of  $s_{=k}^\alpha$  for  $1 < k < \frac{1}{\alpha} + 1$ . This case is not discussed much in (Lemmens and Seidel, 1973) except in Lemma 9 (Lemmens and Seidel, 1973, Theorem 4.4). Our idea is to generalize the decomposition (4) for any  $k$  ( $1 < k < \frac{1}{\alpha} + 1$ ). Again, by the definition of  $E_{=k}^\alpha$ , for any  $X \in E_{=k}^\alpha$ , there exists  $P = \{p_1, \dots, p_k\} \subset X$  such that for any  $i \neq j$ ,  $\langle p_i, p_j \rangle = -\alpha$ . We remark that in this case ( $1 < k < \frac{1}{\alpha} + 1$ ),  $p_1, \dots, p_k$  are linearly independent. We say such  $P$  is a  $k$ -base of  $X$ .

Now for any  $X \in E_{=k}^\alpha$ , for a given  $k$ -base  $P$  of  $X$  and for any  $i$  ( $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$ ), define subsets of  $X$  with respect to  $P$  and  $\xi_{k,i}$ , say  $P_{k,i}$  and  $P_{k,i}^-$ :

$$P_{k,i} \triangleq \bigcup_{j=1}^{b_{k,i}} P_{k,i,j}, \quad P_{k,i}^- \triangleq \bigcup_{j=1}^{b_{k,i}} P_{k,i,j}^-$$



where

$$\begin{aligned} P_{k,i,j} &\triangleq \{x \in X \mid (\langle x, p_1 \rangle, \dots, \langle x, p_k \rangle) = \alpha \epsilon_{k,i,j}\} \\ P_{k,i,j}^- &\triangleq \{x \in X \mid (\langle x, p_1 \rangle, \dots, \langle x, p_k \rangle) = -\alpha \epsilon_{k,i,j}\} \end{aligned}$$

Directly, we have a partition of  $X$ :

$$X = P \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left( P_{k,i} \bigcup P_{k,i}^- \right) = P \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} \left( P_{k,i,j} \bigcup P_{k,i,j}^- \right). \quad (5)$$

Lemmas 5–6 below help us to remove “ $P_{k,i}^-$ ” (or “ $P_{k,i,j}^-$ ”) from the decomposition (5). For any  $X \in E_{=k}^\alpha$ , if there exists a  $k$ -base  $P$  of  $X$  such that  $P_{k,i}^- = \emptyset$  for all  $i$ , then we say  $P$  is a *positive  $k$ -base* of  $X$ . Define  $E_{=k}^{\alpha,+} \triangleq \{X \in E_{=k}^\alpha \mid X \text{ has a positive } k\text{-base}\}$ . Denote  $\max_{X \in E_{=k}^{\alpha,+}} |X|$  by  $s_{=k}^{\alpha,+}$ .

**Lemma 5.**  $s_{=k}^\alpha \leq \max(s_{\geq k+1}^\alpha, s_{=k}^{\alpha,+})$ .

*Proof.* For any  $X \in E_{=k}^\alpha$ , suppose  $X$  has the representation (5) for some  $k$ -base  $P$ . Let

$$Y = P \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor} \left( P_{k,i} \bigcup \left( \bigcup_{x \in P_{k,i}^-} \{-x\} \right) \right).$$

Then  $|Y| = |X|$  and  $Y$  either belongs to  $E_{\geq k+1}^\alpha$  or belongs to  $E_{=k}^{\alpha,+}$ .  $\square$

**Remark 3.** *The construction of  $Y$  in the proof of Lemma 5 is related to the idea of switching equivalence. That is, when making the correspondence between equiangular lines and equiangular vectors, one may always negate vectors and still have the same set of lines structurally.*

**Lemma 6.**  $s^\alpha \leq \max\left(r, s_{=2}^{\alpha,+}, \dots, s_{=\frac{1}{\alpha}}^{\alpha,+}, s_{=\frac{1}{\alpha}+1}^\alpha\right)$ .

*Proof.* The conclusion follows from Lemma 2, Lemma 3, and Lemma 5.  $\square$

Lemma 6 says that we can focus on  $s_{=k}^{\alpha,+}$  in order to solve an upper bound for  $s^\alpha$ . For any  $X \in E_{=k}^{\alpha,+}$ , suppose  $P$  is a positive  $k$ -base of  $X$ . Then the decomposition (5) becomes

$$X = P \bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j}. \quad (6)$$

**Lemma 7.** *For any  $X \in E_{=k}^{\alpha,+}$  and for any positive  $k$ -base  $P$  of  $X$ ,  $P_{k,0} = \emptyset$ .*

*Proof.* If  $P_{k,0} \neq \emptyset$ , then for any  $x \in P_{k,0}$ ,  $(\langle x, p_1 \rangle, \dots, \langle x, p_k \rangle)$  is  $(-\alpha, \dots, -\alpha)$ . So  $\{x, p_1, \dots, p_k\} \in E_{\geq k+1}^\alpha$  and hence  $X \in E_{\geq k+1}^\alpha$ , which contradicts to the hypothesis that  $X \in E_{=k}^{\alpha,+}$ .  $\square$

By Lemma 7, the decomposition (6) becomes

$$X = P \bigcup_{i=1}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j}. \quad (7)$$

Note each  $P_{k,i,j}$  in (7) is a subset of a pillar  $h_{k,i,j} + \Pi_P^\perp$  where

$$h_{k,i,j} \triangleq a^{(1)}p_1 + \dots + a^{(k)}p_k, \text{ and } (a^{(1)}, \dots, a^{(k)})^T = \alpha \left( (1+\alpha)I_k - \alpha J_k \right)^{-1} \epsilon_{k,i,j}^T. \quad (8)$$

By (8), we compute by elementary linear algebra that

$$\langle h_{k,i,j}, h_{k,i,j} \rangle = \alpha^2 \frac{4\alpha i^2 - 4\alpha i k + k(1+\alpha)}{(1+\alpha)(1-(k-1)\alpha)}. \quad (9)$$

Note  $\langle h_{k,i,j}, h_{k,i,j} \rangle$  does not depend on  $j$ . That means for any  $x \in P_{k,i} = \bigcup_{j=1}^{b_{k,i}} P_{k,i,j}$ , the length of its projection on  $\Pi_P$  is uniform. We define  $\beta_{k,i} \triangleq \langle h_{k,i,j}, h_{k,i,j} \rangle$ . In the rest of this subsection, we derive explicit/computable upper bound for  $|P_{k,i,j}|$ , see Lemma 11 and Corollary 3.

**Lemma 8.** *For any  $k = 2, \dots, \frac{1}{\alpha}$ , for any  $i = 1, \dots, \lfloor \frac{k}{2} \rfloor$ , we have*

$$\begin{cases} 0 \leq \beta_{k,i} < \alpha, & i > k - \frac{\frac{1}{\alpha}+1}{2} \\ \beta_{k,i} = \alpha, & i = k - \frac{\frac{1}{\alpha}+1}{2} \\ \alpha < \beta_{k,i} < 1, & i < k - \frac{\frac{1}{\alpha}+1}{2} \end{cases}$$

*Proof.* Directly,  $\beta_{k,i} = \langle h_{k,i,j}, h_{k,i,j} \rangle \geq 0$ . By (9), if  $i = k - \frac{\frac{1}{\alpha}+1}{2}$ , then  $\beta_{k,i} = \alpha$ . Consider  $\beta_{k,i} - \alpha$  as a quadratic function of  $i$ . This function is strictly decreasing when  $0 < i < \frac{k}{2}$  and reaches the unique local minimum when  $i = \frac{k}{2}$ . So  $\beta_{k,i} < \alpha$  if  $k - \frac{\frac{1}{\alpha}+1}{2} < i \leq \frac{k}{2}$  and  $\beta_{k,i} > \alpha$  if  $1 \leq i < k - \frac{\frac{1}{\alpha}+1}{2}$ . The last step is to prove  $\beta_{k,i} < 1$ . Equivalently, we show

$$g(k, i) \triangleq \alpha^2 (4\alpha i^2 - 4\alpha i k + k(1+\alpha)) - (1+\alpha)(1-(k-1)\alpha) < 0.$$

In fact,  $g(\frac{1}{\alpha}, 0) = 0$ . Note  $g(\frac{1}{\alpha}, i)$  is strictly decreasing with respect to  $i$  if  $0 < i < \frac{1}{\alpha}$ . So  $g(\frac{1}{\alpha}, i) < 0$  for any  $1 \leq i \leq \lfloor \frac{1}{2\alpha} \rfloor$ . Now fix an  $i$  ( $1 \leq i \leq \lfloor \frac{1}{2\alpha} \rfloor$ ), then  $g(k, i)$  is strictly increasing with respect to  $k$ . So for any  $2 \leq k \leq \frac{1}{\alpha} - 1$  and for any  $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ ,  $g(k, i) < g(\frac{1}{\alpha}, i) < 0$ .  $\square$

**Lemma 9** ((Lemmens and Seidel, 1973)). *Suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ . For any  $i$  ( $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ ) and for any  $j$  ( $1 \leq j \leq b_{k,i}$ ),  $G(P_{k,i,j}) - \beta_{k,i}J_s$  is a positive semidefinite matrix with rank at most  $r - k$ , where  $s \triangleq |P_{k,i,j}|$ .*

*Proof.* Let  $P_{k,i,j} = \{x_1, \dots, x_s\}$ . For  $t = 1, \dots, s$ , assume  $x_t = h_{k,i,j} + c_t$  where  $c_t \in \Pi_P^\perp$ . Denote  $\{c_1, \dots, c_s\}$  by  $C$ . Then we have  $G(C) = G(P_{k,i,j}) - \langle h_{k,i,j}, h_{k,i,j} \rangle J_s$ . Note that  $\langle h_{k,i,j}, h_{k,i,j} \rangle = \beta_{k,i}$ . Thus,  $G(C) = G(P_{k,i,j}) - \beta_{k,i}J_s$  and hence  $G(P_{k,i,j}) - \beta_{k,i}J_s$  is a positive semidefinite matrix with rank at most  $r - k$ .  $\square$

**Lemma 10.** *For any  $X \in E_{=k}^{\alpha,+}$  and for any positive  $k$ -base  $P$  of  $X$ , we have*

$$P_{k,i,j} \in E^\alpha \setminus E_{\geq i+1}^\alpha, \quad \forall i = 1, \dots, \lfloor \frac{k}{2} \rfloor \text{ and } \forall j = 1, \dots, b_{k,i}.$$

*Proof.* Assume there exist  $x_1, \dots, x_{i+1} \in P_{k,i,j}$  such that  $\langle x_{t_1}, x_{t_2} \rangle = -\alpha$  for any  $1 \leq t_1 < t_2 \leq i+1$ . By the definition of  $P_{k,i,j}$ , there exist  $k-i$  vectors in  $P$ , say  $p_1, \dots, p_{k-i}$ , such that for any  $n = 1, \dots, k-i$  and for any  $t = 1, \dots, i+1$ ,  $\langle p_n, x_t \rangle = -\alpha$ . So the  $k+1$  vectors  $p_1, \dots, p_{k-i}, x_1, \dots, x_{i+1}$  have pairwise inner product  $-\alpha$  and hence  $X \in E_{\geq k+1}^\alpha$ , which contradicts to the hypothesis that  $X \in E_{=k}^{\alpha,+} \subset E_k^\alpha$ .  $\square$

**Corollary 2.** For any  $X \in E_{=k}^{\alpha,+}$ , for any positive  $k$ -base  $P$  of  $X$ , and for all  $j = 1, \dots, b_{k,1}$ ,

$$\langle x_1, x_2 \rangle = \alpha, \forall x_1, x_2 \in P_{k,1,j} \ (x_1 \neq x_2).$$

*Proof.* By Lemma 10,  $P_{k,1,j} \notin E_{\geq 2}^\alpha$ . By the definition of  $E_{\geq 2}^\alpha$ , we directly see that for any two different  $x_1, x_2 \in P_{k,1,j}$ ,  $\langle x_1, x_2 \rangle = \alpha$ .  $\square$

**Lemma 11.** Suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ . For all  $j = 1, \dots, b_{k,1}$ ,

$$|P_{k,1,j}| \leq \begin{cases} r - k, & 1 \geq k - \frac{\frac{1}{\alpha} + 1}{2} \\ \frac{1 - \alpha}{\beta_{k,1} - \alpha}, & 1 < k - \frac{\frac{1}{\alpha} + 1}{2} \end{cases}$$

*Proof.* Let  $s = |P_{k,1,j}|$ . By the Corollary 2, we know all the off-diagonal entries of  $G(P_{k,1,j})$  are  $\alpha$ . So

$$G(P_{k,1,j}) - \beta_{k,1} J_s = (1 - \beta_{k,1}) \left( \left( 1 + \frac{\beta_{k,1} - \alpha}{1 - \beta_{k,1}} \right) I_s - \frac{\beta_{k,1} - \alpha}{1 - \beta_{k,1}} J_s \right). \quad (10)$$

By Lemma 8, if  $1 \geq k - \frac{\frac{1}{\alpha} + 1}{2}$ , then  $\beta_{k,1} \leq \alpha$ . Now it is seen from (10) that  $G(P_{k,1,j}) - \beta_{k,1} J_s$  is positive definite and hence is full rank. So, by Lemma 9, we have  $s \leq r - k$ . By Lemma 8, if  $1 < k - \frac{\frac{1}{\alpha} + 1}{2}$ , then  $\beta_{k,1} > \alpha$ . Note  $1 - \beta_{k,1}$  is positive. Now all off-diagonal entries of  $\frac{1}{1 - \beta_{k,1}} (G(P_{k,1,j}) - \beta_{k,1} J_s)$  are  $-\frac{\beta_{k,1} - \alpha}{1 - \beta_{k,1}}$ , which is negative. By Lemma 9,  $G(P_{k,1,j}) - \beta_{k,1} J_s$  is positive semidefinite and hence all its eigenvalues are positive. By checking these eigenvalues, we have  $s \leq \frac{1}{\frac{\beta_{k,1} - \alpha}{1 - \beta_{k,1}}} + 1 = \frac{1 - \alpha}{\beta_{k,1} - \alpha}$ .  $\square$

**Lemma 12.** Suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ . If  $i > 1$ , then for any  $j = 1, \dots, b_{k,i}$ ,

$$|P_{k,i,j}| \leq s^{\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}} (r)$$

*Proof.* Let  $s = |P_{k,i,j}|$ . Note the positive semidefinite matrix  $M \triangleq \frac{1}{1 - \beta_{k,i}} (G(P_{k,i,j}) - \beta_{k,i} J_s)$  has 1's along the diagonal and two different numbers  $\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}$  or  $\frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}$  as the off-diagonal entries. So  $M$  is the Gramian matrix of a spherical two-distance set  $\{c_1, \dots, c_s\}$  such that

$$\langle c_t, c_m \rangle = \begin{cases} 1 & t = m \\ \frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}} \text{ or } \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} & t \neq m \end{cases}$$

Therefore,  $s \leq s^{\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}} (r)$ .  $\square$

**Corollary 3.** Suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ . If  $i > 1$ , then for any  $j = 1, \dots, b_{k,i}$ ,

$$|P_{k,i,j}| \leq \begin{cases} r + 1, & i < k - \frac{\frac{1}{\alpha} + 1}{2} \\ r - k + \lfloor 2\alpha \frac{r - k}{1 - \alpha} \rfloor, & i = k - \frac{\frac{1}{\alpha} + 1}{2} \\ \text{sdp} \left( r, \frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} \right), & i > k - \frac{\frac{1}{\alpha} + 1}{2} \end{cases}$$

*Proof.* By Lemma 12 and Theorem 1, we directly have

$$|P_{k,i,j}| \leq \mathbf{sdp} \left( r, \frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} \right).$$

Particularly, if  $i < k - \frac{\frac{1}{\alpha} + 1}{2}$  or  $i = k - \frac{\frac{1}{\alpha} + 1}{2}$ , we can derive an explicit bound below.

(i) If  $i < k - \frac{\frac{1}{\alpha} + 1}{2}$ , then by Lemma 8,  $\beta_{k,i} > \alpha$ . Note  $1 - \beta_{k,i} > 0$ . So both  $\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}$  and  $\frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}$  are negative. By (Barg and Yu, 2013, Theorem 3.2, last case),  $|P_{k,i,j}| \leq r + 1$ .

(ii) If  $i = k - \frac{\frac{1}{\alpha} + 1}{2}$ , then by Lemma 8,  $\beta_{k,i} = \alpha$ . So  $\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}} = 0$  and  $\frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} = \frac{-2\alpha}{1 - \alpha}$ . By (Lemmens and Seidel, 1973, Theorem 4.1),  $|P_{k,i,j}| \leq r - k + \lfloor 2\alpha \frac{r-k}{1-\alpha} \rfloor$ .  $\square$

Corollary 3 makes the upper bound of  $|P_{k,i,j}|$  computable. Remark that for the case (ii) in the proof of Corollary 3, experiments show that the explicit bound  $r - k + \lfloor 2\alpha \frac{r-k}{1-\alpha} \rfloor$  is generally smaller than the SDP bound  $\mathbf{sdp} \left( r, 0, \frac{-2\alpha}{1-\alpha} \right)$ . Note  $\beta_{k,i}$  is a rational function in  $\alpha, k, i$ , see formula (9). So the right side of the last case in (3), namely  $\mathbf{sdp} \left( r, \frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} \right)$ , can be computed by solving SDP if  $r, \alpha, k, i$  are concrete.

## 4. Main results

### 4.1. A general method for computing an upper bound of $s(r)$

Now we are prepared to present a general method, namely Theorem 4, for computing an upper bound of  $s(r)$  for any  $r > 3$ . The proof of Theorem 4 is mainly based on the decomposition (7), Lemma 11 and Corollary 3. Note again an upper bound of  $s^{\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}}(r)$  in Theorem 4 can be computed by  $\mathbf{sdp} \left( r, \frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}} \right)$ .

**Theorem 4.** For any  $r > 3$ , suppose  $K$  is the largest positive integer such that  $(2K - 1)^2 \leq 2r$ . Then an upper bound of  $s(r)$  can be computed by (A)(B)(C)(D) below.

$$(A) \quad s(r) \leq \max \left( 2r + 3, s^{\frac{1}{5}}(r), \dots, s^{\frac{1}{2K-1}}(r) \right).$$

$$(B) \quad \text{For } \alpha = \frac{1}{5}, \dots, \frac{1}{2K-1}, s^\alpha(r) \leq \max \left( r, s_{=2}^{\alpha,+}(r), \dots, s_{=\frac{1}{\alpha}}^{\alpha,+}(r), s_{=\frac{1}{\alpha}+1}^\alpha(r) \right).$$

$$(C) \quad \text{For } \alpha = \frac{1}{5}, \dots, \frac{1}{2K-1}, \text{ for } k = \frac{1}{\alpha} + 1, s_{=k}^\alpha \leq k + \frac{1}{2} \binom{k}{\frac{k}{2}} \left( r - k + 1 + \lfloor 2\alpha \frac{r-k+1}{1-\alpha} \rfloor \right).$$

$$(D) \quad \text{For } \alpha = \frac{1}{5}, \dots, \frac{1}{2K-1}, \text{ for } k = 2, \dots, \frac{1}{\alpha}, \text{ we have the upper bound for } s_{=k}^{\alpha,+}(r) \text{ below where } N(\alpha) \triangleq \frac{1+3\alpha}{2\alpha}.$$

$$(a) \quad s_{=2}^{\alpha,+}(r) \leq r.$$

(b) If  $2 < k \leq N(\alpha)$ , then the upper bound of  $s_{=k}^{\alpha,+}(r)$  is

$$k + k(r - k) + \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} b_{k,i} s^{\frac{\alpha - \beta_{k,i}}{1 - \beta_{k,i}}, \frac{-\alpha - \beta_{k,i}}{1 - \beta_{k,i}}}(r). \quad (11)$$

(c) If  $N(\alpha) < k \leq \frac{1}{\alpha}$ , then the upper bound of  $s_{=k}^{\alpha,+}(r)$  is

$$k + \frac{k(1-\alpha)}{\beta_{k,1}-\alpha} + \sum_{i=2}^{k-N(\alpha)} b_{k,i} (r+1) + b_{k,k-N(\alpha)+1} \left( r - k + \lfloor 2\alpha \frac{r-k}{1-\alpha} \rfloor \right) + \sum_{i=k-N(\alpha)+2}^{\lfloor \frac{k}{2} \rfloor} b_{k,i} s^{\frac{\alpha-\beta_{k,i}}{1-\beta_{k,i}}, \frac{-\alpha-\beta_{k,i}}{1-\beta_{k,i}}}(r). \quad (12)$$

*Proof.* Note that (A), (B) and (C) are Corollary 1, Lemma 6, and Theorem 3, respectively. Now we only need to prove (D). For each fixed  $k$ , suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ .

(a) If  $k = 2$ , then  $\lfloor \frac{k}{2} \rfloor = 1$ . Note  $b_{2,1} = \frac{1}{2} \binom{2}{1} = 1$ . So the formula (7) becomes  $X = P \cup P_{2,1,1}$ . By Lemma 11,  $|P_{2,1,1}| \leq r - 2$ . So  $|X| \leq |P| + |P_{2,1,1}| = 2 + r - 2 = r$ . And hence  $s_{=k}^{\alpha,+} \leq r$ .

(b) If  $2 < k \leq N(\alpha)$ , note  $k \leq N(\alpha) \Leftrightarrow 1 \geq k - \frac{1+\alpha}{2}$ . By Lemma 11,  $|P_{k,1,j}| \leq r - k$ . Note  $b_{k,1} = \binom{k}{1} = k$ . So  $\left| \bigcup_{j=1}^{b_{k,1}} P_{k,1,j} \right| \leq k(r - k)$ . For any  $i > 1$ ,  $1 \geq k - \frac{1+\alpha}{2}$  implies  $i > k - \frac{1+\alpha}{2}$ . So by Corollary 3,

$$\left| \bigcup_{i=2}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j} \right| \leq \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} b_{k,i} s^{\frac{\alpha-\beta_{k,i}}{1-\beta_{k,i}}, \frac{-\alpha-\beta_{k,i}}{1-\beta_{k,i}}}(r)$$

Therefore, by (7),

$$|X| \leq |P| + \left| \bigcup_{j=1}^{b_{k,1}} P_{k,1,j} \right| + \left| \bigcup_{i=2}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j} \right| \leq k + k(r - k) + \sum_{i=2}^{\lfloor \frac{k}{2} \rfloor} b_{k,i} s^{\frac{\alpha-\beta_{k,i}}{1-\beta_{k,i}}, \frac{-\alpha-\beta_{k,i}}{1-\beta_{k,i}}}(r),$$

and hence (11) is proved.

(c) If  $k > N(\alpha)$ , note  $k > N(\alpha) \Leftrightarrow 1 < k - \frac{1+\alpha}{2}$ . By Lemma 11,  $|P_{k,1,j}| \leq \frac{(1-\alpha)}{\beta_{k,1}-\alpha}$ . So  $\left| \bigcup_{j=1}^{b_{k,1}} P_{k,1,j} \right| \leq \frac{k(1-\alpha)}{\beta_{k,1}-\alpha}$ . By Corollary 3,

$$\begin{aligned} \left| \bigcup_{i=2}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j} \right| &= \left| \bigcup_{i=2}^{k-N(\alpha)} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j} \right| + \left| \bigcup_{j=1}^{b_{k,k-N(\alpha)+1}} P_{k,k-N(\alpha)+1,j} \right| + \left| \bigcup_{i=k-N(\alpha)+2}^{\lfloor \frac{k}{2} \rfloor} \bigcup_{j=1}^{b_{k,i}} P_{k,i,j} \right| \\ &\leq \sum_{i=2}^{k-N(\alpha)} b_{k,i} (r+1) + b_{k,k-N(\alpha)+1} \left( r - k + \lfloor 2\alpha \frac{r-k}{1-\alpha} \rfloor \right) + \sum_{i=k-N(\alpha)+2}^{\lfloor \frac{k}{2} \rfloor} b_{k,i} s^{\frac{\alpha-\beta_{k,i}}{1-\beta_{k,i}}, \frac{-\alpha-\beta_{k,i}}{1-\beta_{k,i}}}(r). \end{aligned}$$

And hence we have (12).  $\square$

#### 4.2. Relative bound for $s^{\frac{1}{\alpha}}(r)$

The goal of this subsection is to present a computable relative bound for  $s^{\frac{1}{\alpha}}(r)$ , see Theorem 5. Theorem 4(B–D) plays the heart role in the proof of Theorem 5. Particularly, the upper bound for  $s_{=k}^{\alpha,+}$  given in Theorem 4(D) can be further improved when  $k = \frac{1+3\alpha}{2\alpha}$ , see Lemma 13 and Lemma 14. Note also that Theorem 4(B–D) and Lemma 13 provides a general method for computing relative bounds for any  $\alpha$ . So the proof of Theorem 5 can be trivially generalized for any  $s^\alpha(r)$ .

First, we define the notation

$$\gamma_{k,i_1,j_1,i_2,j_2} \triangleq \langle h_{k,i_1,j_1}, h_{k,i_2,j_2} \rangle, \text{ where } i_1 \neq i_2 \text{ or } j_1 \neq j_2.$$

Note that  $\gamma_{k,i_1,j_1,i_2,j_2}$  can be computed easily by (8) when concrete  $i_1, i_2, j_1, j_2$  are given. Lemma 13 is inspired by (Lemmens and Seidel, 1973, Theorem 4.3).

**Lemma 13.** For  $k = \frac{1+3\alpha}{2\alpha}$ , suppose  $X \in E_{=k}^{\alpha,+}$  and  $P$  is a positive  $k$ -base of  $X$ . If both  $P_{k,1,j_1}, P_{k,i,j_2}$  are non-empty for some  $i, j_1$  and  $j_2$  ( $i \neq 1$  or  $j_1 \neq j_2$ ), then

$$|P_{k,1,j_1}| \leq \lfloor \frac{(1-\alpha)(1-\beta_{k,i})}{(\alpha-\gamma_{k,1,j_1,i,j_2})^2} \rfloor \quad (13)$$

*Proof.* Suppose  $P_{k,1,j_1} = \{x_1, \dots, x_s\}$  and  $x_{s+1} \in P_{k,i,j_2}$ . For  $t = 1, \dots, s$ , assume  $x_t = h_{k,1,j_1} + c_t$  and assume  $x_{s+1} = h_{k,i,j_2} + c_{s+1}$  where  $c_t \in \Pi_P^\perp$  ( $1 \leq t \leq s+1$ ). Denote  $\{c_1, \dots, c_s, c_{s+1}\}$  by  $C$ . Note  $k = \frac{1+3\alpha}{2\alpha}$ . By Lemma 8,  $\langle h_{k,1,j_1}, h_{k,1,j_1} \rangle = \beta_{k,1} = \alpha$ . Note also  $\langle h_{k,i,j_2}, h_{k,i,j_2} \rangle = \beta_{k,i}$  and  $\langle h_{k,1,j_1}, h_{k,i,j_2} \rangle = \gamma_{k,1,j_1,i,j_2}$ . Then we have

$$G(C) = \left( \begin{array}{c|c} (1-\alpha)I_s & z^T \\ \hline z & 1-\beta_{k,i} \end{array} \right)_{(s+1) \times (s+1)}$$

where  $z$  is a vector in  $\mathbb{R}^s$  and each coordinate of  $z$  is either  $\alpha - \gamma_{k,1,j_1,i,j_2}$  or  $-\alpha - \gamma_{k,1,j_1,i,j_2}$ . By the fact that  $G(C)$  is positive semidefinite, we derive  $s \leq \lfloor \frac{(1-\alpha)(1-\beta_{k,i})}{(\alpha-\gamma_{k,1,j_1,i,j_2})^2} \rfloor$  (see the last step of the proof of (Lemmens and Seidel, 1973, Theorem 4.3)).  $\square$

**Lemma 14.**

$$s_{=5}^{\frac{1}{7},+} \leq \max \left( 725, r, 195 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r), 301 + 4 s^{\frac{1}{19}, -\frac{5}{19}}(r), 407 + s^{\frac{1}{19}, -\frac{5}{19}}(r) \right) \quad (14)$$

*Proof.* Note for  $\alpha = \frac{1}{7}$  and  $k = 5$ , we have  $k = \frac{1+3\alpha}{2\alpha}$ . Now we derive an upper bound for  $s_{=5}^{\frac{1}{7},+}$  by Lemma 13. Suppose  $X \in E_{=5}^{\frac{1}{7},+}$  and  $P$  is a positive 5-base of  $X$ . By (7),

$$X = P \cup P_{5,1} \cup P_{5,2} = P \cup \left( \bigcup_{j=1}^5 P_{5,1,j} \right) \cup \left( \bigcup_{j=1}^{10} P_{5,2,j} \right)$$

- (i) If  $P_{5,1} = \emptyset$  and  $P_{5,2} = \emptyset$ , then  $|X| = |P| = 5$ .
- (ii) If  $P_{5,1} \neq \emptyset$  and  $P_{5,2} = \emptyset$ , then by Lemma 11 and Lemma 13,

$$|X| \leq \begin{cases} 5 + r - 5 = r, & \text{if only one } P_{5,1,j} \text{ is non-empty} \\ 5 + 5 \times 144 = 725, & \text{if at least two } P_{5,1,j_1}, P_{5,1,j_2} \text{ are non-empty} \end{cases}$$

- (iii) If  $P_{5,1} = \emptyset$  and  $P_{5,2} \neq \emptyset$ , then by Corollary 3, we have

$$|X| \leq 5 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r).$$

- (vi) If  $P_{5,1} \neq \emptyset$  and  $P_{5,2} \neq \emptyset$ , suppose some  $P_{5,1,j_1}$  and  $P_{5,i,j_2}$  ( $i \neq 1$  or  $j_1 \neq j_2$ ) are non-empty. By checking the different values of the right side of (13) for different choices of  $i, j_1, j_2$  in (13), we have the three facts below:

- (F1) if  $i = 1$ , then  $|P_{5,1,j_1}| \leq 144$ ;

–(F2) if  $i = 2$  and there is one coordinate where both  $\epsilon_{5,1,j_1}$  and  $\epsilon_{5,2,j_2}$  take the value 1 (for instance,  $\epsilon_{5,1,j_1} = (1, -1, -1, -1, -1)$  and  $\epsilon_{5,2,j_2} = (1, 1, -1, -1, -1)$ ), then  $|P_{5,1,j_1}| \leq 152$ ;

–(F3) if  $i = 2$  and  $\epsilon_{5,1,j_1}$  and  $\epsilon_{5,2,j_2}$  do not take the value 1 at the same coordinate, then  $|P_{5,1,j_1}| \leq 38$ .

Based on (F1)(F2)(F3), we have conclusions as follows. If  $P_{5,2,j}$  is non-empty for all  $1 \leq j \leq 10$ , then,

$$|X| \leq 5 + 5 \times 38 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r) = 195 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r). \quad (15)$$

If only one  $P_{5,2,j}$  is non-empty, then,

$$|X| \leq 5 + 2 \times 144 + 3 \times 38 + s^{\frac{1}{19}, -\frac{5}{19}}(r) = 407 + s^{\frac{1}{19}, -\frac{5}{19}}(r).$$

Without loss of generality, assume that there is at least one coordinate where the first four vectors in  $\xi_{5,2}$  take the value 1. If any  $P_{5,2,j}$  ( $1 \leq j \leq 4$ ) is non-empty and any other  $P_{5,2,j}$  ( $j > 4$ ) is empty, then,

$$|X| \leq 5 + 144 + 4 \times 38 + 4 s^{\frac{1}{19}, -\frac{5}{19}}(r) = 301 + 4 s^{\frac{1}{19}, -\frac{5}{19}}(r). \quad (16)$$

Finally, one can check that if the number of non-empty  $P_{5,2,j}$  is between 2 and 4, then  $|X|$  is bounded by (16) and if the number of non-empty  $P_{5,2,j}$  is between 5 and 9, then  $|X|$  is bounded by (15). Above all, by (i–vi), we conclude (14).  $\square$

**Theorem 5.** *The upper bound of  $s^{\frac{1}{7}}(r)$  is*

$$\begin{aligned} & \max \left( 3r - 6, 4r - 12 + 3 s^{\frac{1}{13}, -\frac{3}{13}}(r), \right. \\ & \quad \min \left( 5r - 20 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r), \right. \\ & \quad \quad \max \left( 725, r, 195 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r), 301 + 4 s^{\frac{1}{19}, -\frac{5}{19}}(r), 407 + s^{\frac{1}{19}, -\frac{5}{19}}(r) \right), \\ & \quad \left. 15r - 36 + 15 \left\lfloor \frac{r-6}{3} \right\rfloor + 10 s^{\frac{1}{25}, -\frac{7}{25}}(r), 56r - 203 + 35 \left\lfloor \frac{r-7}{3} \right\rfloor \right) \end{aligned}$$

*Proof.* By Theorem 4 (B),

$$s^{\frac{1}{7}} \leq \max \left( r, \max_{2 \leq k \leq 7} s_{=k}^{\frac{1}{7}, +}, s_{=8}^{\frac{1}{7}} \right). \quad (17)$$

By Theorem 4(C), we have

$$s_{=8}^{\frac{1}{7}, +} \leq 35r - 237 + 35 \left\lfloor \frac{r-7}{3} \right\rfloor. \quad (18)$$

By Theorem 4(D)(a–b), we have

$$\max \left( s_{=2}^{\frac{1}{7}, +}, s_{=3}^{\frac{1}{7}, +} \right) \leq \max(r, 3r - 6) = 3r - 6 \text{ (notice } r > 3), \quad (19)$$

$$s_{=4}^{\frac{1}{7}, +} \leq 4r - 12 + 3 s^{\frac{1}{13}, -\frac{3}{13}}(r), \quad (20)$$

$$s_{=5}^{\frac{1}{7}, +} \leq 5r - 20 + 10 s^{\frac{1}{19}, -\frac{5}{19}}(r). \quad (21)$$

By Theorem 4 (c), we have

$$s_{=6}^{\frac{1}{7},+} \leq 15r - 36 + 15 \lfloor \frac{r-6}{3} \rfloor + 10 s_{=5}^{\frac{1}{25},-\frac{7}{25}}(r), \quad (22)$$

$$s_{=7}^{\frac{1}{7},+} \leq 56r - 203 + 35 \lfloor \frac{r-7}{3} \rfloor. \quad (23)$$

Note Lemma 14 provides another upper bound for  $s_{=5}^{\frac{1}{7},+}$ , see (14). By (17–23) and (14), the conclusion is proved.  $\square$

#### 4.3. Relative bound for $s^{\frac{1}{5}}(r)$

The goal of this subsection is to present a computable relative bound for  $s^{\frac{1}{5}}(r)$ , see Theorem 6. Besides applying Theorem 4(B–D) and Lemma 13 as what we have done in the proof of Theorem 5, we further improve the upper bound for  $s_{=5}^{\frac{1}{5}}$  by the Lemma 15 below.

**Lemma 15.**  $s_{=5}^{\frac{1}{5}} \leq \max(280, r + 5 + \lfloor \frac{1}{2}(r-5) \rfloor)$ .

*Proof.* Suppose  $X \in E_{=5}^{\frac{1}{5},+}$  and  $P = \{p_1, \dots, p_5\}$  is a positive 5-base of  $X$ . By (7),

$$X = P \cup P_{5,1} \cup P_{5,2} = P \cup \left( \bigcup_{j=1}^5 P_{5,1,j} \right) \cup \left( \bigcup_{j=1}^{10} P_{5,2,j} \right)$$

We first prove two claims.

(C1)  $|P_{5,1}| \leq 5$ .

In fact, by Lemma 11, for any  $j$  ( $1 \leq j \leq 5$ ),  $|P_{5,1,j}| \leq 3$ . If two different  $P_{5,1,j_1}, P_{5,1,j_2}$  are non-empty, take  $x_1, x_2, x_3 \in P_{5,1,j_1}$  and  $x_4 \in P_{5,1,j_2}$ . Suppose  $x_i = h_{5,1,j_1} + c_i$  ( $i = 1, 2, 3$ ) and  $x_4 = h_{5,1,j_2} + c_4$  where  $c_i \in \Pi_P^\perp$ . We compute that

$$G(c_1, c_2, c_3, c_4) = \frac{8}{15} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & g_{14} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & g_{24} \\ -\frac{1}{2} & -\frac{1}{2} & 1 & g_{34} \\ g_{14} & g_{24} & g_{34} & 1 \end{pmatrix}$$

where  $g_{i4}$  ( $i = 1, 2, 3$ ) is either  $-\frac{1}{4}$  or  $-1$ . However, one can check that the matrix above is never positive semidefinite for all the possible choices of  $g_{i4}$ . So when two different  $P_{5,1,j_1}, P_{5,1,j_2}$  are non-empty, there are at most 2 vectors in each  $P_{5,1,j}$  ( $j = j_1, j_2$ ). Similarly, one can check that the only way to have strictly more than 4 vectors in total is to fill only 1 vector  $x_j$  in each  $P_{5,1,j}$  ( $j = 1, 2, 3, 4, 5$ ) and the pairwise inner product of these 5 vectors is  $\frac{1}{5}$ . So (C1) is proved.

(C2)  $|P \cup P_{5,2}| \leq \max(275, r + \lfloor \frac{1}{2}(r-5) \rfloor)$ .

In fact, let  $p_6 = -\sum_{i=1}^5 p_i$ . Then it is seen that  $\{p_6\} \cup P \cup P_{5,2} \in E_{=6}^{\frac{1}{5}}$ . So by Lemma 4,

$$|\{p_6\} \cup P \cup P_{5,2}| \leq \max\left(276, r + 1 + \lfloor \frac{1}{2}(r-5) \rfloor\right).$$

Hence (C2) is proved.



By (C1)(C2), we have  $s_{=5}^{\frac{1}{5},+} \leq \max(280, r + 5 + \lfloor \frac{1}{2}(r - 5) \rfloor)$ . Finally by Lemma 5 and Lemma 4, we have

$$s_{=5}^{\frac{1}{5}} \leq \max\left(s_{=5}^{\frac{1}{5},+}, s_{=6}^{\frac{1}{5}}\right) \leq \max\left(280, r + 5 + \lfloor \frac{1}{2}(r - 5) \rfloor\right).$$

□

**Theorem 6.** *The upper bound of  $s^{\frac{1}{5}}(r)$  is*

$$\max\left(3r - 6, 280, r + 5 + \lfloor \frac{1}{2}(r - 5) \rfloor, \min\left(4r - 12 + 3 s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r), \max\left(148, r, 95 + s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r), 69 + 3 s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r)\right)\right)\right)$$

*Proof.* By Theorem 4(B),

$$s_{=5}^{\frac{1}{5}} \leq \max\left(r, \max_{2 \leq k \leq 5} s_{=k}^{\frac{1}{5},+}, s_{=6}^{\frac{1}{5}}\right). \quad (24)$$

By Theorem 4 (D)(a–b), we have

$$\max\left(s_{=2}^{\frac{1}{5},+}, s_{=3}^{\frac{1}{5},+}\right) \leq \max(r, 3r - 6) = 3r - 6 \text{ (notice } r > 3), \quad (25)$$

$$s_{=4}^{\frac{1}{5},+} \leq 4r - 12 + 3 s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r). \quad (26)$$

By Lemma 4 and Lemma 15, we have

$$\max\left(s_{=5}^{\frac{1}{5}}, s_{=6}^{\frac{1}{5}}\right) \leq \max\left(280, r + 5 + \lfloor \frac{1}{2}(r - 5) \rfloor\right). \quad (27)$$

Note for  $\alpha = \frac{1}{5}$  and  $k = 4$ , we have  $k = \frac{1+3\alpha}{2\alpha}$ . By applying Lemma 13 and the similar discuss as the proof of Lemma 14, we have

$$s_{=4}^{\frac{1}{5},+} \leq \max\left(148, r, 95 + s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r), 69 + 3 s_{=4}^{\frac{1}{13}, -\frac{5}{13}}(r)\right). \quad (28)$$

Combining (24–28), the conclusion is proved. □

**Remark 4.** *The last second sentence in the paragraph before the (Lemmens and Seidel, 1973, Conjecture 5.8) states without proof that*

$$s_{=k}^{\frac{1}{5}} \leq \max\left(276, r + 1 + \lfloor \frac{1}{2}(r - 5) \rfloor\right), \text{ for } k = 2, 3, 5. \quad (29)$$

*Unfortunately, we have not been able to completely verify this. If (29) is true, then we can remove “ $3r - 6$ ” from Theorem 6 and replace “ $280, r + 5 + \lfloor \frac{1}{2}(r - 5) \rfloor$ ” with “ $276, r + 1 + \lfloor \frac{1}{2}(r - 5) \rfloor$ ” respectively in Theorem 6. However, it does not affect our main result Theorem 7 in Section 4.4.*

#### 4.4. Explicit upper bound for $44 \leq r \leq 400$

Now we are prepared to show our main result.

**Theorem 7.** For  $44 \leq r \leq 400$ , the upper bound of maximum number of equiangular lines in  $\mathbb{R}^r$  is

$$\begin{cases} \frac{4r(m+1)(m+2)}{(2m+3)^2-r}, & r = 44, 45, 46, 76, 77, 78, 117, 118, 166, 222, 286, 358 \\ \frac{((2m+1)^2-2)((2m+1)^2-1)}{2}, & \text{other } r \text{ between 44 and 400} \end{cases} \quad (30)$$

and if the upper bound in (30) can be attained, then the relative angle is

$$\begin{cases} \frac{1}{2m+3}, & r = 44, 45, 46, 76, 77, 78, 117, 118, 166, 222, 286, 358 \\ \frac{1}{2m+1}, & \text{other } r \text{ between 44 and 400} \end{cases} \quad (31)$$

where  $m$  is the largest positive integer such that  $(2m+1)^2 \leq r+2$ .

*Proof.* For any  $44 \leq r \leq 400$ , suppose  $K$  is the largest positive integer such that  $(2K-1)^2 \leq 2r$ . For each  $r$ , by Theorem 1(A), we compute the upper bound of  $s^\alpha$  for each  $\alpha = \frac{1}{5}, \frac{1}{7}, \dots, \frac{1}{2K-1}$  and then pick up the maximum  $\max(2r+3, s^{\frac{1}{5}}, \dots, s^{\frac{1}{2K-1}})$ . We summarize the computational results in Table 2 (see the completed computational upper bounds for all  $s^\alpha(r)$  in Appendix A, Table 3).

Table 2: Upper bound of  $s(r)$  for  $44 \leq r \leq 400$

$r$	44	45	46	47–75	76	77	
$s(r) \leq$	422	540	736	1128	1216	1540	
$\operatorname{argmax}_{\alpha} s^{\alpha}(r)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{9}$	
$r$	78	79–116	117	118	119–165	166	
$s(r) \leq$	2080	3160	3510	4720	7140	9296	
$\operatorname{argmax}_{\alpha} s^{\alpha}(r)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{13}$	
$r$	167–221	222	223–285	286	287–357	358	359–400
$s(r) \leq$	14028	16576	24976	27456	41328	42960	64620
$\operatorname{argmax}_{\alpha} s^{\alpha}(r)$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{19}$	$\frac{1}{19}$

More specifically, for  $s^{\frac{1}{5}}$ , we compute the upper bound presented in Theorem 6 and the SDP bound  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  given in Theorem 1, respectively and then we pick up the smaller one of these two upper bounds. For  $s^{\frac{1}{7}}$ , we compute the upper bound presented in Theorem 5 and the SDP bound  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$ , respectively and then we pick up the smaller one of these two upper bounds. For  $\alpha = \frac{1}{9}, \dots, \frac{1}{2K-1}$ , we compute the SDP bound  $\mathbf{sdp}(r, \alpha, -\alpha)$ . One can check Table 2 is equivalent to (30–31).  $\square$

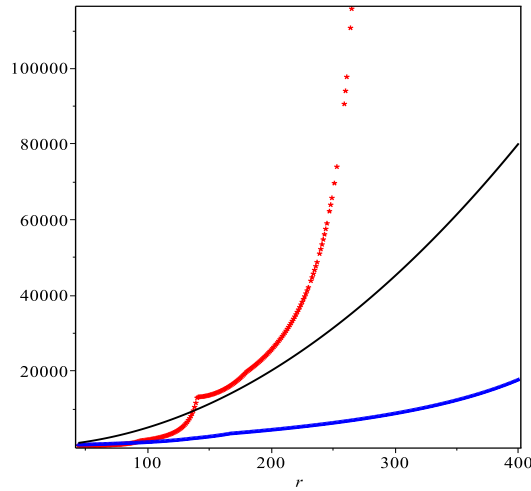
Remark that more computations can be carried out for  $r > 400$ . For those large  $r$ 's, in order to get a non-trivial relative bound of  $s^\alpha(r)$  for  $\alpha \leq \frac{1}{9}$ , one can apply Theorem 4(B–D) and Lemma 13 like what we have done for  $\alpha = \frac{1}{7}$ . According to Table 1, Theorem 7 and our further experiments, we propose a conjecture below.

**Conjecture 1.** For any  $r$ , if  $m$  is the largest positive integer such that  $(2m + 1)^2 \leq r + 2$ , then the upper bound of maximum number of equiangular lines in  $\mathbb{R}^r$  is either  $\frac{4r(m+1)(m+2)}{(2m+3)^2-r}$  or  $\frac{((2m+1)^2-2)((2m+1)^2-1)}{2}$ .

## 5. Experiments

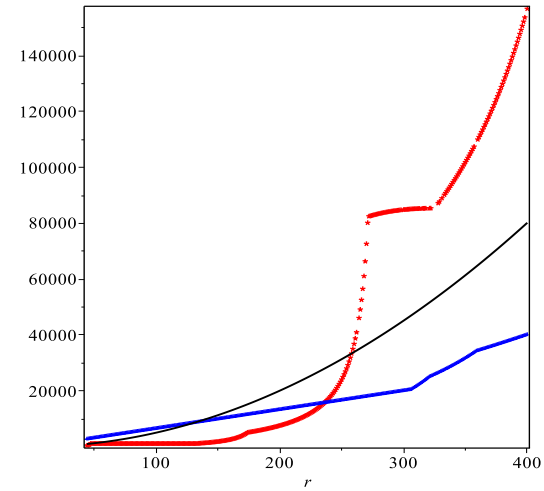
We compare the relative bounds given by Theorem 6 and  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  in Figure 1 and the relative bounds given by Theorem 5 and  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$  in Figure 2.

Figure 1: Comparing Theorem 6 and  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$



— :  $\frac{r(r+1)}{2}$     \*\*\* :  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$     +++ : Theorem 6

Figure 2: Comparing Theorem 5 and  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$



— :  $\frac{r(r+1)}{2}$     \*\*\* :  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$     +++ : Theorem 5

We now explain how to generate these two figures. For Figure 1, for each  $r$  between 44 and 400, we compute the upper bound of  $s^{\frac{1}{5}}(r)$  by Theorem 6 and the SDP bound  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$ , respectively. Note again that  $s^{\frac{1}{13}, -\frac{5}{13}}(r)$  in Theorem 6 is generally not known. So instead of  $s^{\frac{1}{13}, -\frac{5}{13}}(r)$ , we compute  $\mathbf{sdp}(r, \frac{1}{13}, -\frac{5}{13})$  because  $\mathbf{sdp}(r, \frac{1}{13}, -\frac{5}{13})$  is an upper bound of  $s^{\frac{1}{13}, -\frac{5}{13}}(r)$  by Theorem 1. In Figure 1, we mark by blue plusses “+” the computed upper bound according to Theorem 6 and we mark  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  by red stars “\*”. We also draw Gerzon’s bound as a black curve. Similarly, we generate Figure 2 by comparing the relative bounds for  $s^{\frac{1}{7}}(r)$ .

By Figures 1–2, for  $44 \leq r \leq 400$ , the SDP bound is smaller when  $r$  is small, it increases dramatically at some  $r$  and it eventually goes beyond the Gerzon’s bound; the bound by Theorem 6/Theorem 5 is larger when  $r$  is small but it increases slower and when  $r$  is sufficiently large, it always gives non-trivial upper bound which is much smaller than either the SDP bound or the Gerzon’s bound. Some complementary data is as follows (see Appendix A for the concrete data).

- Concerning Figure 1, when  $44 \leq r \leq 84$ ,  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  is smaller than the bound in Theorem 6. When  $85 \leq r \leq 400$ , the bound in Theorem 6 is smaller (generally much smaller) than

$\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$ . One can check the concrete bounds for  $s^{\frac{1}{5}}(r)$  ( $85 \leq r$ ) in Table 3 and compare them with the SDP bounds shown in the red parentheses. The SDP bounds in the red parentheses which we computed are the same as the data shown in Barg and Yu (2014), which is to be expected. As an example of Theorem 6 outperforming SDP, the bound given by Theorem 6 for  $s^{\frac{1}{5}}(137)$  is 2043 while the SDP bound is 9528. When  $137 \leq r \leq 400$ ,  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  is greater (eventually much greater) than Gerzon's bound, which is consistent with the results obtained in (Barg and Yu, 2014). On the other hand, for any  $44 \leq r \leq 400$ , the bound in Theorem 6 is smaller than Gerzon's bound. For a range of larger  $r$ , for instance  $r = 266$ –400 and some discrete  $r$  such as 231, 238 and so on, **sdpt3** failed to compute  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$ . That is the reason why the red markers in Figure 1 are not as continuous as the blue markers.

- Similarly, we note for Figure 2 that when  $44 \leq r \leq 235$ ,  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$  is smaller than the bound in Theorem 5. When  $235 \leq r \leq 400$ , the bound in Theorem 5 is smaller (generally much smaller) than  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$ . When  $259 \leq r \leq 400$ ,  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$  is greater (eventually much greater) than Gerzon's bound. However, for  $130 \leq r \leq 400$ , the bound in Theorem 5 is smaller than Gerzon's bound. For some large  $r$ , **sdpt3** failed to compute  $\mathbf{sdp}(r, \frac{1}{7}, -\frac{1}{7})$ .

In conclusion, practically, for  $\alpha = \frac{1}{5}$  or  $\frac{1}{7}$  and for any  $r$ , we can compute both upper bounds for  $s^\alpha(r)$ : the SDP bound and the bound based on Theorem 6 (respectively, Theorem 5), and then pick up the smaller one. That is how we generate the upper bounds for  $s^{\frac{1}{5}}(r)$ ,  $s^{\frac{1}{7}}(r)$  in Table 3, see the columns " $\frac{1}{5}$ " and " $\frac{1}{7}$ " there.

An extra remark is that for  $\alpha < \frac{1}{7}$ , experiments show that the SDP bounds for  $s^\alpha(r)$  are always no greater than the Gerzon's bound for  $44 \leq r \leq 400$ . See the SDP bounds for  $s^{\frac{1}{9}}(r), \dots, s^{\frac{1}{27}}(r)$  in the columns " $\frac{1}{9}$ ",  $\dots$ , " $\frac{1}{27}$ " in Table 3. A possible reason is that 400 is not large enough for the SDP bound of  $s^\alpha(r)$  ( $r \leq 400, \alpha < \frac{1}{7}$ ) to go beyond the Gerzon's bound as  $\mathbf{sdp}(r, \frac{1}{5}, -\frac{1}{5})$  does at  $r = 137$ .

Either computing Theorem 6 (Theorem 5) or computing SDP bounds, we need to solve SDP. In our experiments, we solve SDP by the Matlab software **CVX 3.0 beta** (Grant and Boyd, 2014, 2008). There are many SDP solvers provided by **CVX 3.0 beta** and the computational results presented in this paper are computed by **sdpt3** (Toh et al., 1999; Tutuncu et al., 2003). The computation is carried out by a 3.20GHz Interl(R) Core(TM) i5-4460 processor under x86\_64 GNU/Linux. Our Matlab version is **R2016a**. The output of **sdpt3** is a floating number. We take the nearest integer to this floating number and record it in Tables 2–3.

## 6. Conclusion and future work

We develop a computable upper bound of the number of equiangular lines in the Euclidean vector space  $\mathbb{R}^r$  by taking the classical pillar decomposition (Lemmens and Seidel, 1973), generalizing it, and combining it with the semidefinite programming (SDP) method (Barg and Yu, 2013, 2014). Based on this computable bound, we improve the relative bounds for  $s^{\frac{1}{5}}$  and  $s^{\frac{1}{7}}$ . The computational results show an explicit and non-trivial bound, which is strictly less than the well-known Gerzon's bound for a general  $r$  ( $44 \leq r \leq 400$ ). Our main result leads us to focus on some special dimensions between 44 to 400: 44, 45, 46, 76, 77, 78, 117, 118, 166, 222, 286, 358 and  $(2m+1)^2 - 2$  (47, 79, 119, 167, 223, 287, 359). We particularly hope that for  $(2m+1)^2 - 2$ , Gerzon's bound can be proven definitively to be saturated or not. Then by our main result,  $s(r)$  will be solved for a

large range of  $r$ . It is known that  $s(47) < 1128$  and  $s(79) < 3160$ , and it is also mentioned in (Barg and Yu, 2014) that  $s(119)$  has been open for a while. We remark that in the first case of the formula (30), the upper bound shown there is exactly the relative bound given in (2) for  $\alpha = \frac{1}{2m+3}$ . It may also be seen that in the second case of the formula (30), the upper bound is exactly Gerzon's bound for  $r = (2m + 1)^2 - 2$ . It would be interesting to explore the reason why the computational method gives such answers. Our proposal for future work is

- studying the spectrahedral shadow of the SDP constraints,
- studying the SDP method from the angle of harmonic analysis in (Bachoc and Vallentin, 2008) and seeing the possibility for modifying the SDP formulation, and
- generalizing methods of equiangular tight frame constructions to find sets of equiangular lines to prove lower bounds on  $s(r)$ .

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## Appendix A

Table 3: Upper bound of number of equiangular lines in  $\mathbb{R}^r$  ( $44 \leq r \leq 400$ )

$r$	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{17}$	$\frac{1}{19}$	$\frac{1}{21}$	$\frac{1}{23}$	$\frac{1}{25}$	$\frac{1}{27}$	max	$\frac{r(r+1)}{2}$	angle
44	276	422	95										422	990	$\frac{1}{2}$
45	276	540	100										540	1035	$\frac{1}{2}$
46	276	736	105										736	1081	$\frac{1}{2}$
47	276	1128	111										1128	1128	$\frac{1}{2}$
48	276	1128	116										1128	1176	$\frac{1}{2}$
49	276	1128	123										1128	1225	$\frac{1}{2}$
50	276	1128	129										1128	1275	$\frac{1}{2}$
51	276	1128	136										1128	1326	$\frac{1}{2}$
52	276	1128	143										1128	1378	$\frac{1}{2}$
53	276	1128	151										1128	1431	$\frac{1}{2}$
54	276	1128	160										1128	1485	$\frac{1}{2}$
55	276	1128	169										1128	1540	$\frac{1}{2}$
56	276	1128	179										1128	1596	$\frac{1}{2}$
57	276	1128	190										1128	1653	$\frac{1}{2}$
58	276	1128	202										1128	1711	$\frac{1}{2}$
59	276	1128	215										1128	1770	$\frac{1}{2}$
60	276	1128	229										1128	1830	$\frac{1}{2}$
61	279	1128	244	122									1128	1891	$\frac{1}{2}$
62	290	1128	261	126									1128	1953	$\frac{1}{2}$
63	301	1128	280	130									1128	2016	$\frac{1}{2}$
64	313	1128	301	135									1128	2080	$\frac{1}{2}$
65	326	1128	325	139									1128	2145	$\frac{1}{2}$
66	339	1128	352	144									1128	2211	$\frac{1}{2}$
67	353	1128	383	149									1128	2278	$\frac{1}{2}$
68	368	1128	418	154									1128	2346	$\frac{1}{2}$
69	383	1128	460	159									1128	2415	$\frac{1}{2}$
70	399	1128	509	165									1128	2485	$\frac{1}{2}$
71	416	1128	568	170									1128	2556	$\frac{1}{2}$
72	434	1128	640	176									1128	2628	$\frac{1}{2}$
73	453	1128	730	183									1128	2701	$\frac{1}{2}$
74	473	1128	846	189									1128	2775	$\frac{1}{2}$
75	495	1128	1000	196									1128	2850	$\frac{1}{2}$
76	518	1128	1216	203									1216	2926	$\frac{1}{10}$
77	542	1128	1540	210									1540	3003	$\frac{1}{10}$
78	568	1128	2080	218									2080	3081	$\frac{1}{10}$
79	596	1128	3160	226									3160	3160	$\frac{1}{10}$
80	626	1128	3160	234									3160	3240	$\frac{1}{10}$
81	658	1128	3160	243									3160	3321	$\frac{1}{10}$
82	693	1128	3160	252									3160	3403	$\frac{1}{10}$
83	731	1128	3160	262									3160	3486	$\frac{1}{10}$
84	772	1128	3160	272									3160	3570	$\frac{1}{10}$
85	789 (817)	1128	3160	283	170								3160	3655	$\frac{1}{10}$
86	804 (866)	1128	3160	295	174								3160	3741	$\frac{1}{10}$
87	819 (920)	1128	3160	307	178								3160	3828	$\frac{1}{10}$
88	831 (980)	1128	3160	320	183								3160	3916	$\frac{1}{10}$
89	846 (1046)	1128	3160	334	187								3160	4005	$\frac{1}{10}$
90	864 (1120)	1128	3160	348	191								3160	4095	$\frac{1}{10}$
91	879 (1204)	1128	3160	364	196								3160	4186	$\frac{1}{10}$
92	894 (1298)	1128	3160	381	201								3160	4278	$\frac{1}{10}$
93	909 (1406)	1128	3160	399	206								3160	4371	$\frac{1}{10}$
94	927 (1515)	1128	3160	418	211								3160	4465	$\frac{1}{10}$
95	942 (1557)	1128	3160	438	216								3160	4560	$\frac{1}{10}$
96	960 (1600)	1128	3160	461	221								3160	4656	$\frac{1}{10}$
97	978 (1645)	1128	3160	485	226								3160	4753	$\frac{1}{10}$
98	993 (1691)	1128	3160	511	232								3160	4851	$\frac{1}{10}$
99	1011 (1740)	1128	3160	540	238								3160	4950	$\frac{1}{10}$
100	1029 (1790)	1128	3160	571	243								3160	5050	$\frac{1}{10}$
101	1047 (1843)	1128	3160	606	250								3160	5151	$\frac{1}{10}$
102	1068 (1897)	1128	3160	644	256								3160	5253	$\frac{1}{10}$
103	1086 (1955)	1128	3160	687	262								3160	5356	$\frac{1}{10}$
104	1107 (2014)	1128	3160	734	269								3160	5460	$\frac{1}{10}$

105	1125 (2077)	1128	3160	788	276									3160	5565	10
106	1146 (2142)	1128	3160	848	283									3160	5671	10
107	1167 (2211)	1128	3160	917	290									3160	5778	10
108	1188 (2283)	1128	3160	997	297									3160	5886	10
109	1209 (2358)	1128	3160	1090	305									3160	5995	10
110	1230 (2438)	1128	3160	1200	313									3160	6105	10
111	1254 (2521)	1128	3160	1332	322									3160	6216	10
112	1275 (2609)	1128	3160	1493	330									3160	6328	10
113	1299 (2702)	1128	3160	1695	339	226								3160	6441	10
114	1323 (2801)	1128	3160	1954	348	230								3160	6555	10
115	1347 (2905)	1128	3160	2300	358	234								3160	6670	10
116	1374 (3015)	1128	3160	2784	368	238								3160	6786	10
117	1398 (3132)	1128	3160	3510	378	243								3510	6903	11
118	1425 (3257)	1128	3160	4720	389	247								4720	7021	11
119	1452 (3390)	1128	3160	7140	400	251								7140	7140	11
120	1479 (3532)	1128	3160	7140	411	256								7140	7260	11
121	1506 (3684)	1128	3160	7140	424	261								7140	7381	11
122	1533 (3848)	1128	3160	7140	436	265								7140	7503	11
123	1563 (4024)	1128	3160	7140	449	270								7140	7626	11
124	1593 (4214)	1128	3160	7140	463	275								7140	7750	11
125	1623 (4420)	1128	3160	7140	477	280								7140	7875	11
126	1653 (4643)	1128	3160	7140	492	285								7140	8001	11
127	1686 (4887)	1128	3160	7140	508	290								7140	8128	11
128	1719 (5154)	1128	3160	7140	524	296								7140	8256	11
129	1752 (5447)	1128	3160	7140	542	301								7140	8385	11
130	1788 (5771)	1128	3160	7140	560	307								7140	8515	11
131	1821 (6130)	1128	3160	7140	579	312								7140	8646	11
132	1857 (6532)	1130	3160	7140	599	318								7140	8778	11
133	1896 (6983)	1158	3160	7140	621	324								7140	8911	11
134	1932 (7493)	1188	3160	7140	643	330								7140	9045	11
135	1971 (8076)	1218	3160	7140	667	336								7140	9180	11
136	2010 (8747)	1249	3160	7140	692	342								7140	9316	11
137	2043 (9529)	1282	3160	7140	719	349								7140	9453	11
138	2073 (10450)	1316	3160	7140	748	355								7140	9591	11
139	2106 (11553)	1351	3160	7140	778	362								7140	9730	11
140	2136 (12896)	1387	3160	7140	811	369								7140	9870	11
141	2169 (13118)	1425	3160	7140	846	376								7140	10011	11
142	2202 (13141)	1465	3160	7140	884	383								7140	10153	11
143	2238 (13172)	1506	3160	7140	924	391								7140	10296	11
144	2271 (13212)	1549	3160	7140	968	398								7140	10440	11
145	2307 (13260)	1594	3160	7140	1015	406	290							7140	10585	11
146	2343 (13316)	1640	3160	7140	1066	414	294							7140	10731	11
147	2382 (13380)	1689	3160	7140	1123	422	298							7140	10878	11
148	2418 (13451)	1740	3160	7140	1184	431	302							7140	11026	11
149	2457 (13529)	1794	3160	7140	1252	439	307							7140	11175	11
150	2496 (13615)	1850	3160	7140	1326	448	311							7140	11325	11
151	2535 (13708)	1909	3160	7140	1409	457	315							7140	11476	11
152	2577 (13808)	1971	3160	7140	1502	466	320							7140	11628	11
153	2619 (13914)	2037	3160	7140	1607	476	324							7140	11781	11
154	2661 (14028)	2105	3160	7140	1725	486	329							7140	11935	11
155	2703 (14149)	2178	3160	7140	1860	496	333							7140	12090	11
156	2748 (14276)	2255	3160	7140	2016	506	338							7140	12246	11
157	2793 (14411)	2336	3160	7140	2198	517	343							7140	12403	11
158	2838 (14552)	2422	3160	7140	2413	528	347							7140	12561	11
159	2886 (14701)	2514	3160	7140	2671	540	352							7140	12720	11
160	2934 (14856)	2612	3160	7140	2987	551	357							7140	12880	11
161	2982 (15019)	2716	3160	7140	3381	564	362							7140	13041	11
162	3033 (15190)	2827	3160	7140	3888	576	367							7140	13203	11
163	3087 (15368)	2946	3160	7140	4564	589	373							7140	13366	11
164	3138 (15554)	3074	3160	7140	5510	602	378							7140	13530	11
165	3195 (15747)	3212	3160	7140	6930	616	383							7140	13695	11
166	3252 (15949)	3361	3160	7140	9296	630	389							9296	13861	13
167	3309 (16160)	3522	3160	7140	14028	645	394							14028	14028	13
168	3336 (16379)	3697	3160	7140	14028	660	400							14028	14196	13
169	3363 (16606)	3889	3160	7140	14028	676	406							14028	14365	13
170	3390 (16844)	4098	3160	7140	14028	692	411							14028	14535	13



171	3420 (17090)	4328	3160	7140	14028	709	417							14028	14706	17
172	3447 (17347)	4583	3160	7140	14028	727	423							14028	14878	17
173	3474 (17614)	4866	3160	7140	14028	745	430							14028	15051	17
174	3504 (17891)	5127	3160	7140	14028	764	436							14028	15225	17
175	3531 (18180)	5199	3160	7140	14028	784	442							14028	15400	17
176	3561 (18480)	5273	3160	7140	14028	805	449							14028	15576	17
177	3588 (18793)	5348	3160	7140	14028	826	455							14028	15753	17
178	3618 (19118)	5425	3160	7140	14028	848	462							14028	15931	17
179	3645 (19457)	5503	3160	7140	14028	872	469							14028	16110	17
180	3675 (19799)	5582	3160	7140	14028	896	476							14028	16290	17
181	3705 (20035)	5664	3160	7140	14028	921	483	362						14028	16471	17
182	3735 (20276)	5747	3160	7140	14028	948	490	366						14028	16653	17
183	3762 (20522)	5832	3160	7140	14028	976	497	370						14028	16836	17
184	3792 (20773)	5918	3160	7140	14028	1005	505	374						14028	17020	17
185	3822 (21029)	6006	3160	7140	14028	1036	512	378						14028	17205	17
186	3852 (21290)	6097	3160	7140	14028	1068	520	383						14028	17391	17
187	3882 (21557)	6189	3160	7140	14028	1102	528	387						14028	17578	17
188	3912 (21830)	6283	3160	7140	14028	1138	536	391						14028	17766	17
189	3942 (22108)	6380	3160	7140	14028	1176	544	396						14028	17955	17
190	3972 (22393)	6478	3160	7140	14028	1216	553	400						14028	18145	17
191	4005 (22683)	6579	3160	7140	14028	1258	561	404						14028	18336	17
192	4035 (22980)	6682	3160	7140	14028	1303	570	409						14028	18528	17
193	4065 (23284)	6787	3160	7140	14028	1351	579	414						14028	18721	17
194	4098 (23594)	6895	3160	7140	14028	1402	588	418						14028	18915	17
195	4128 (23911)	7005	3160	7140	14028	1456	597	423						14028	19110	17
196	4158 (24236)	7118	3160	7140	14028	1514	607	428						14028	19306	17
197	4191 (24567)	7234	3160	7140	14028	1576	617	432						14028	19503	17
198	4224 (24907)	7352	3160	7140	14028	1643	627	437						14028	19701	17
199	4254 (25254)	7473	3160	7140	14028	1714	637	442						14028	19900	17
200	4287 (25609)	7598	3160	7140	14028	1792	647	447						14028	20100	17
201	4320 (25973)	7725	3160	7140	14028	1876	658	452						14028	20301	17
202	4350 (26346)	7856	3160	7140	14028	1967	669	457						14028	20503	17
203	4383 (26727)	7990	3160	7140	14028	2067	680	463						14028	20706	17
204	4416 (27118)	8128	3160	7140	14028	2176	691	468						14028	20910	17
205	4449 (27519)	8269	3160	7140	14028	2296	703	473						14028	21115	17
206	4482 (27929)	8414	3160	7140	14028	2429	715	478						14028	21321	17
207	4515 (28350)	8563	3160	7140	14028	2576	727	484						14028	21528	17
208	4548 (28782)	8716	3160	7140	14028	2741	740	489						14028	21736	17
209	4584 (29225)	8874	3160	7140	14028	2926	752	495						14028	21945	17
210	4617 (29679)	9036	3160	7140	14028	3136	766	501						14028	22155	17
211	4650 (30146)	9202	3160	7140	14028	3376	779	506						14028	22366	17
212	4686 (30625)	9373	3160	7140	14028	3653	793	512						14028	22578	17
213	4719 (31116)	9550	3160	7140	14028	3976	807	518						14028	22791	17
214	4755 (31622)	9731	3160	7140	14028	4358	822	524						14028	23005	17
215	4788 (32141)	9918	3160	7140	14028	4816	837	530						14028	23220	17
216	4824 (32675)	10111	3160	7140	14028	5376	852	536						14028	23436	17
217	4857 (33224)	10310	3160	7140	14028	6076	868	543						14028	23653	17
218	4893 (33790)	10516	3160	7140	14028	6976	884	549						14028	23871	17
219	4929 (34371)	10728	3160	7140	14028	8176	901	555						14028	24090	17
220	4965 (34970)	10947	3160	7140	14028	9856	918	562						14028	24310	17
221	5001 (35587)	11173	3160	7140	14028	12376	936	568	442					14028	24531	17
222	5037 (36223)	11407	3160	7140	14028	16576	954	575	446					16576	24753	17
223	5073 (36879)	11649	3160	7140	14028	24976	973	582	450					24976	24976	17
224	5109 (37555)	11899	3160	7140	14028	24976	992	589	454					24976	25200	17
225	5145 (38253)	12159	3160	7140	14028	24976	1013	596	458					24976	25425	17
226	5184 (38974)	12428	3160	7140	14028	24976	1033	603	463					24976	25651	17
227	5220 (39718)	12706	3160	7140	14028	24976	1054	610	467					24976	25878	17
228	5256 (40486)	12996	3185	7140	14028	24976	1076	617	471					24976	26106	17
229	5295 (41283)	13296	3245	7140	14028	24976	1099	625	475					24976	26335	17
230	5331 (42106)	13609	3306	7140	14028	24976	1123	632	480					24976	26565	17
231	5370 (NaN)	13934	3370	7140	14028	24976	1147	640	484					24976	26796	17
232	5409 (43840)	14272	3435	7140	14028	24976	1172	647	488					24976	27028	17
233	5448 (44755)	14624	3502	7140	14028	24976	1198	655	493					24976	27261	17
234	5484 (45703)	14991	3571	7140	14028	24976	1225	663	497					24976	27495	17
235	5523 (46688)	15374	3643	7140	14028	24976	1253	671	502					24976	27730	17
236	5562 (47710)	15673 (15775)	3716	7140	14028	24976	1282	680	507					24976	27966	17

237	5601 (48773)	15729 (16193)	3792	7140	14028	24976	1313	688	511				24976	28203	$\frac{1}{15}$
238	5643 (NaN)	15820 (16631)	3871	7140	14028	24976	1344	697	516				24976	28441	$\frac{1}{15}$
239	5682 (51028)	15876 (17090)	3952	7140	14028	24976	1377	705	521				24976	28680	$\frac{1}{15}$
240	5721 (52226)	15932 (17571)	4036	7140	14028	24976	1411	714	525				24976	28920	$\frac{1}{15}$
241	5760 (53475)	16023 (18076)	4123	7140	14028	24976	1446	723	530				24976	29161	$\frac{1}{15}$
242	5802 (54778)	16079 (18607)	4213	7140	14028	24976	1483	732	535				24976	29403	$\frac{1}{15}$
243	5841 (56139)	16135 (19167)	4306	7140	14028	24976	1521	741	540				24976	29646	$\frac{1}{15}$
244	5883 (57562)	16226 (19756)	4402	7140	14028	24976	1562	751	545				24976	29890	$\frac{1}{15}$
245	5925 (59051)	16282 (20378)	4502	7140	14028	24976	1604	760	550				24976	30135	$\frac{1}{15}$
246	5964 (NaN)	16338 (21036)	4606	7140	14028	24976	1648	770	555				24976	30381	$\frac{1}{15}$
247	6006 (62245)	16429 (21732)	4714	7140	14028	24976	1694	780	560				24976	30628	$\frac{1}{15}$
248	6048 (63963)	16485 (22471)	4826	7140	14028	24976	1742	790	565				24976	30876	$\frac{1}{15}$
249	6090 (65768)	16541 (23255)	4942	7140	14028	24976	1793	800	571				24976	31125	$\frac{1}{15}$
250	6132 (NaN)	16632 (24091)	5063	7140	14028	24976	1846	811	576				24976	31375	$\frac{1}{15}$
251	6177 (69668)	16688 (24982)	5190	7140	14028	24976	1902	821	581				24976	31626	$\frac{1}{15}$
252	6219 (NaN)	16744 (25934)	5321	7140	14028	24976	1962	832	587				24976	31878	$\frac{1}{15}$
253	6261 (74013)	16835 (26955)	5459	7140	14028	24976	2024	843	592				24976	32131	$\frac{1}{15}$
254	6306 (NaN)	16891 (28051)	5602	7140	14028	24976	2090	855	598				24976	32385	$\frac{1}{15}$
255	6348 (NaN)	16947 (29230)	5752	7140	14028	24976	2160	866	603				24976	32640	$\frac{1}{15}$
256	6393 (NaN)	17038 (30506)	5909	7140	14028	24976	2234	878	609				24976	32896	$\frac{1}{15}$
257	6435 (NaN)	17094 (31887)	6073	7140	14028	24976	2313	890	615				24976	33153	$\frac{1}{15}$
258	6480 (NaN)	17150 (33388)	6246	7140	14028	24976	2397	902	620				24976	33411	$\frac{1}{15}$
259	6525 (90616)	17241 (35025)	6427	7140	14028	24976	2486	914	626				24976	33670	$\frac{1}{15}$
260	6570 (94071)	17297 (36818)	6617	7140	14028	24976	2582	927	632				24976	33930	$\frac{1}{15}$
261	6615 (97777)	17353 (38791)	6818	7140	14028	24976	2685	940	638				24976	34191	$\frac{1}{15}$
262	6660 (NaN)	17444 (40971)	7029	7140	14028	24976	2795	953	644				24976	34453	$\frac{1}{15}$
263	6705 (NaN)	17500 (NaN)	7252	7140	14028	24976	2913	966	650				24976	34716	$\frac{1}{15}$
264	6753 (110739)	17556 (46100)	7488	7140	14028	24976	3041	980	656				24976	34980	$\frac{1}{15}$
265	6798 (115802)	17647 (49145)	7737	7140	14028	24976	3180	994	662	530			24976	35245	$\frac{1}{15}$
266	6846 (NaN)	17703 (52596)	8002	7140	14028	24976	3331	1008	669	534			24976	35511	$\frac{1}{15}$
267	6891 (NaN)	17759 (56539)	8283	7140	14028	24976	3495	1023	675	538			24976	35778	$\frac{1}{15}$
268	6939 (NaN)	17850 (61089)	8583	7140	14028	24976	3675	1037	682	542			24976	36046	$\frac{1}{15}$
269	6987 (NaN)	17906 (66396)	8902	7140	14028	24976	3874	1053	688	546			24976	36315	$\frac{1}{15}$
270	7035 (NaN)	17962 (72667)	9243	7140	14028	24976	4093	1068	695	550			24976	36585	$\frac{1}{15}$
271	7083 (NaN)	18053 (80191)	9609	7140	14028	24976	4336	1084	701	555			24976	36856	$\frac{1}{15}$
272	7131 (NaN)	18109 (82514)	10002	7140	14028	24976	4608	1100	708	559			24976	37128	$\frac{1}{15}$
273	7179 (NaN)	18165 (82645)	10425	7140	14028	24976	4914	1117	715	563			24976	37401	$\frac{1}{15}$
274	7230 (NaN)	18256 (82772)	10882	7140	14028	24976	5261	1134	722	567			24976	37675	$\frac{1}{15}$
275	7278 (NaN)	18312 (82895)	11377	7140	14028	24976	5657	1151	729	572			24976	37950	$\frac{1}{15}$
276	7329 (NaN)	18368 (83014)	11915	7140	14028	24976	6114	1169	736	576			24976	38226	$\frac{1}{15}$
277	7377 ( $\infty$ )	18459 (83129)	12502	7140	14028	24976	6648	1187	743	580			24976	38503	$\frac{1}{15}$
278	7428 (NaN)	18515 (83240)	13012	7140	14028	24976	7279	1206	750	585			24976	38781	$\frac{1}{15}$
279	7479 (NaN)	18571 (83348)	13122	7140	14028	24976	8035	1225	758	589			24976	39060	$\frac{1}{15}$
280	7530 (NaN)	18662 (83452)	13235	7140	14028	24976	8960	1244	765	594			24976	39340	$\frac{1}{15}$
281	7581 (NaN)	18718 (83552)	13349	7140	14028	24976	10116	1265	773	598			24976	39621	$\frac{1}{15}$
282	7632 (NaN)	18774 (83649)	13464	7140	14028	24976	11602	1285	780	603			24976	39903	$\frac{1}{15}$
283	7683 (NaN)	18865 (83742)	13580	7140	14028	24976	13584	1306	788	607			24976	40186	$\frac{1}{15}$
284	7737 (NaN)	18921 (83832)	13698	7140	14028	24976	16358	1328	796	612			24976	40470	$\frac{1}{15}$
285	7788 (NaN)	18977 (83918)	13817	7140	14028	24976	20520	1350	804	617			24976	40755	$\frac{1}{15}$
286	7842 (NaN)	19068 (84003)	13937	7140	14028	24976	27456	1373	812	621			27456	41041	$\frac{1}{17}$
287	7896 ( $\infty$ )	19124 (84084)	14059	7140	14028	24976	41328	1396	820	626			41328	41328	$\frac{1}{17}$
288	7950 ( $\infty$ )	19180 (84161)	14183	7140	14028	24976	41328	1420	828	631			41328	41616	$\frac{1}{17}$
289	8004 (NaN)	19271 (84236)	14308	7140	14028	24976	41328	1445	837	636			41328	41905	$\frac{1}{17}$
290	8058 ( $\infty$ )	19327 (84307)	14434	7140	14028	24976	41328	1470	845	641			41328	42195	$\frac{1}{17}$
291	8112 ( $\infty$ )	19383 (84376)	14562	7140	14028	24976	41328	1497	854	646			41328	42486	$\frac{1}{17}$
292	8166 ( $\infty$ )	19474 (84441)	14692	7140	14028	24976	41328	1523	862	651			41328	42778	$\frac{1}{17}$
293	8223 ( $\infty$ )	19530 (84505)	14824	7140	14028	24976	41328	1551	871	656			41328	43071	$\frac{1}{17}$
294	8280 ( $\infty$ )	19586 (84565)	14957	7140	14028	24976	41328	1580	880	661			41328	43365	$\frac{1}{17}$
295	8334 ( $\infty$ )	19677 (84623)	15087	7140	14028	24976	41328	1609	889	666			41328	43660	$\frac{1}{17}$
296	8391 ( $\infty$ )	19733 (84678)	15228	7140	14028	24976	41328	1639	898	671			41328	43956	$\frac{1}{17}$
297	8448 ( $\infty$ )	19789 (84731)	15366	7140	14028	24976	41328	1671	907	676			41328	44253	$\frac{1}{17}$
298	8505 ( $\infty$ )	19880 (84781)	15506	7140	14028	24976	41328	1703	917	681			41328	44551	$\frac{1}{17}$
299	8565 ( $\infty$ )	19936 (NaN)	15648	7140	14028	24976	41328	1736	926	686			41328	44850	$\frac{1}{17}$
300	8622 ( $\infty$ )	19992 (84874)	15791	7140	14028	24976	41328	1770	936	692			41328	45150	$\frac{1}{17}$
301	8682 ( $\infty$ )	20083 (84918)	15937	7140	14028	24976	41328	1806	946	697			41328	45451	$\frac{1}{17}$
302	8739 ( $\infty$ )	20139 (84959)	16084	7140	14028	24976	41328	1843	956	702			41328	45753	$\frac{1}{17}$

303	8799 (∞)	20195 (84998)	16233	7140	14028	24976	41328	1881	966	708			41328	46056	17
304	8859 (∞)	20286 (85035)	16385	7140	14028	24976	41328	1920	976	713			41328	46360	17
305	8919 (∞)	20342 (85069)	16538	7140	14028	24976	41328	1961	987	719			41328	46665	17
306	8982 (∞)	20405 (85102)	16694	7140	14028	24976	41328	2003	997	725			41328	46971	17
307	9042 (∞)	20665 (85132)	16851	7140	14028	24976	41328	2047	1008	730			41328	47278	17
308	9105 (∞)	20935 (85161)	17011	7140	14028	24976	41328	2092	1019	736			41328	47586	17
309	9165 (NaN)	21205 (85188)	17173	7140	14028	24976	41328	2139	1030	742			41328	47895	17
310	9228 (∞)	21485 (85212)	17337	7140	14028	24976	41328	2188	1041	747			41328	48205	17
311	9291 (∞)	21775 (85236)	17503	7140	14028	24976	41328	2239	1053	753			41328	48516	17
312	9354 (NaN)	22065 (85257)	17672	7140	14028	24976	41328	2292	1064	759			41328	48828	17
313	9420 (∞)	22365 (85275)	17843	7140	14028	24976	41328	2348	1076	765	626		41328	49141	17
314	9483 (NaN)	22665 (NaN)	18017	7140	14028	24976	41328	2405	1088	771	630		41328	49455	17
315	9549 (∞)	22975 (85310)	18193	7140	14028	24976	41328	2465	1100	777	634		41328	49770	17
316	9612 (NaN)	23295 (85325)	18372	7140	14028	24976	41328	2528	1112	783	638		41328	50086	17
317	9678 (∞)	23615 (85338)	18553	7140	14028	24976	41328	2594	1125	790	642		41328	50403	17
318	9744 (∞)	23945 (85349)	18737	7140	14028	24976	41328	2662	1138	796	646		41328	50721	17
319	9813 (∞)	24285 (NaN)	18923	7140	14028	24976	41328	2734	1150	802	651		41328	51040	17
320	9879 (∞)	24625 (NaN)	19113	7140	14028	24976	41328	2810	1164	808	655		41328	51360	17
321	9948 (NaN)	24985 (85375)	19305	7140	14028	24976	41328	2889	1177	815	659		41328	51681	17
322	10017 (∞)	25175 (85381)	19500	7140	14028	24976	41328	2972	1191	821	663		41328	52003	17
323	10083 (∞)	25375 (NaN)	19698	7140	14028	24976	41328	3060	1204	828	667		41328	52326	17
324	10155 (∞)	25575 (NaN)	19899	7140	14028	24976	41328	3152	1218	834	672		41328	52650	17
325	10224 (∞)	25785 (NaN)	20103	7140	14028	24976	41328	3250	1233	841	676		41328	52975	17
326	10293 (∞)	25985 (NaN)	20310	7140	14028	24976	41328	3353	1247	848	680		41328	53301	17
327	10365 (NaN)	26195 (NaN)	20521	7140	14028	24976	41328	3462	1262	855	685		41328	53628	17
328	10437 (∞)	26405 (87203)	20735	7140	14028	24976	41328	3578	1277	862	689		41328	53956	17
329	10509 (NaN)	26615 (87800)	20952	7140	14028	24976	41328	3701	1292	869	694		41328	54285	17
330	10581 (∞)	26835 (NaN)	21173	7140	14028	24976	41328	3832	1308	876	698		41328	54615	17
331	10653 (∞)	27045 (88997)	21397	7140	14028	24976	41328	3972	1324	883	703		41328	54946	17
332	10728 (∞)	27265 (NaN)	21625	7140	14028	24976	41328	4121	1340	890	707		41328	55278	17
333	10803 (∞)	27485 (90224)	21856	7140	14028	24976	41328	4281	1357	897	712		41328	55611	17
334	10878 (∞)	27715 (90848)	22092	7140	14028	24976	41328	4453	1373	904	716		41328	55945	17
335	10953 (∞)	27935 (91479)	22331	7140	14028	24976	41328	4638	1391	912	721		41328	56280	17
336	11028 (∞)	28165 (92116)	22574	7140	14028	24976	41328	4838	1408	919	725		41328	56616	17
337	11106 (∞)	28395 (92764)	22822	7140	14028	24976	41328	5055	1426	927	730		41328	56953	17
338	11184 (∞)	28625 (93418)	23073	7140	14028	24976	41328	5290	1444	934	735		41328	57291	17
339	11262 (∞)	28865 (94077)	23329	7140	14028	24976	41328	5547	1462	942	740		41328	57630	17
340	11340 (∞)	29105 (94748)	23589	7140	14028	24976	41328	5829	1481	950	744		41328	57970	17
341	11418 (∞)	29345 (95426)	23854	7140	14028	24976	41328	6138	1500	958	749		41328	58311	17
342	11499 (∞)	29595 (96111)	24124	7140	14028	24976	41328	6480	1520	966	754		41328	58653	17
343	11580 (NaN)	29835 (96805)	24398	7140	14028	24976	41328	6860	1540	974	759		41328	58996	17
344	11661 (∞)	30085 (97508)	24678	7140	14028	24976	41328	7285	1560	982	764		41328	59340	17
345	11742 (∞)	30345 (98218)	24962	7140	14028	24976	41328	7763	1581	990	769		41328	59685	17
346	11826 (∞)	30595 (98938)	25252	7140	14028	24976	41328	8304	1603	998	774		41328	60031	17
347	11910 (∞)	30855 (99667)	25546	7140	14028	24976	41328	8923	1624	1007	779		41328	60378	17
348	11991 (∞)	31115 (100403)	25847	7205	14028	24976	41328	9637	1646	1015	784		41328	60726	17
349	12078 (∞)	31385 (101152)	26153	7314	14028	24976	41328	10470	1669	1024	789		41328	61075	17
350	12162 (∞)	31645 (101908)	26464	7426	14028	24976	41328	11455	1692	1032	794		41328	61425	17
351	12249 (∞)	31915 (102674)	26782	7540	14028	24976	41328	12636	1716	1041	799		41328	61776	17
352	12336 (∞)	32195 (103448)	27106	7658	14028	24976	41328	14080	1740	1050	805		41328	62128	17
353	12423 (∞)	32465 (104234)	27436	7778	14028	24976	41328	15885	1765	1059	810		41328	62481	17
354	12510 (∞)	32745 (105030)	27773	7901	14028	24976	41328	18206	1790	1068	815		41328	62835	17
355	12600 (∞)	33035 (105834)	28116	8028	14028	24976	41328	21300	1816	1077	820		41328	63190	17
356	12690 (∞)	33325 (106649)	28466	8158	14028	24976	41328	25632	1843	1087	826		41328	63546	17
357	12780 (∞)	33615 (107477)	28824	8292	14028	24976	41328	32130	1870	1096	831		41328	63903	17
358	12873 (∞)	33915 (NaN)	29188	8429	14028	24976	41328	42960	1898	1105	837		42960	64261	19
359	12966 (∞)	34215 (NaN)	29560	8570	14028	24976	41328	64620	1926	1115	842		64620	64620	19
360	13059 (∞)	34355 (110023)	29940	8715	14028	24976	41328	64620	1956	1125	848		64620	64980	19
361	13152 (∞)	34485 (110892)	30328	8864	14028	24976	41328	64620	1985	1135	853		64620	65341	19
362	13245 (∞)	34625 (111776)	30724	9018	14028	24976	41328	64620	2016	1145	859		64620	65703	19
363	13341 (∞)	34765 (112670)	31129	9176	14028	24976	41328	64620	2048	1155	865		64620	66066	19
364	13437 (∞)	34895 (113577)	31543	9338	14028	24976	41328	64620	2080	1165	870		64620	66430	19
365	13536 (∞)	35035 (114495)	31965	9506	14028	24976	41328	64620	2113	1175	876	730	64620	66795	19
366	13635 (∞)	35175 (115424)	32397	9679	14028	24976	41328	64620	2147	1186	882	734	64620	67161	19
367	13734 (∞)	35315 (116371)	32839	9857	14028	24976	41328	64620	2182	1196	888	738	64620	67528	19
368	13833 (NaN)	35445 (117328)	33291	10041	14028	24976	41328	64620	2218	1207	894	742	64620	67896	19

369	13932 ( $\infty$ )	35585 (118299)	33753	10231	14028	24976	41328	64620	2255	1218	899	746	64620	68265	$\frac{1}{19}$
370	14034 ( $\infty$ )	35725 (119282)	34226	10427	14028	24976	41328	64620	2293	1229	905	750	64620	68635	$\frac{1}{19}$
371	14139 ( $\infty$ )	35865 (120280)	34711	10629	14028	24976	41328	64620	2332	1240	911	754	64620	69006	$\frac{1}{19}$
372	14241 ( $\infty$ )	36005 (121292)	35206	10839	14028	24976	41328	64620	2372	1251	918	759	64620	69378	$\frac{1}{19}$
373	14346 ( $\infty$ )	36145 (122319)	35714	11055	14028	24976	41328	64620	2414	1262	924	763	64620	69751	$\frac{1}{19}$
374	14451 ( $NaN$ )	36285 (123360)	36234	11279	14028	24976	41328	64620	2456	1274	930	767	64620	70125	$\frac{1}{19}$
375	14559 ( $\infty$ )	36425 (124416)	36767	11512	14028	24976	41328	64620	2500	1286	936	771	64620	70500	$\frac{1}{19}$
376	14667 ( $\infty$ )	36565 (125488)	37313	11752	14028	24976	41328	64620	2545	1298	942	775	64620	70876	$\frac{1}{19}$
377	14775 ( $\infty$ )	36715 (126573)	37874	12002	14028	24976	41328	64620	2592	1310	949	780	64620	71253	$\frac{1}{19}$
378	14886 ( $\infty$ )	36855 (127678)	38448	12261	14028	24976	41328	64620	2640	1322	955	784	64620	71631	$\frac{1}{19}$
379	14997 ( $\infty$ )	36995 (128798)	39038	12530	14028	24976	41328	64620	2690	1334	961	788	64620	72010	$\frac{1}{19}$
380	15108 ( $\infty$ )	37135 (129934)	39643	12810	14028	24976	41328	64620	2741	1347	968	793	64620	72390	$\frac{1}{19}$
381	15219 ( $\infty$ )	37285 (131088)	40264	13100	14028	24976	41328	64620	2794	1359	974	797	64620	72771	$\frac{1}{19}$
382	15333 ( $\infty$ )	37425 (132258)	40902	13403	14028	24976	41328	64620	2849	1372	981	801	64620	73153	$\frac{1}{19}$
383	15450 ( $\infty$ )	37565 (133447)	41557	13718	14028	24976	41328	64620	2906	1385	988	806	64620	73536	$\frac{1}{19}$
384	15567 ( $\infty$ )	37715 (134654)	42231	14047	14028	24976	41328	64620	2964	1398	994	810	64620	73920	$\frac{1}{19}$
385	15684 ( $\infty$ )	37855 (135880)	42924	14389	14028	24976	41328	64620	3025	1412	1001	815	64620	74305	$\frac{1}{19}$
386	15801 ( $NaN$ )	38005 (137124)	43637	14748	14028	24976	41328	64620	3088	1425	1008	819	64620	74691	$\frac{1}{19}$
387	15921 ( $\infty$ )	38145 (138388)	44370	15122	14028	24976	41328	64620	3153	1439	1015	824	64620	75078	$\frac{1}{19}$
388	16044 ( $\infty$ )	38295 (139672)	45125	15514	14028	24976	41328	64620	3221	1453	1022	828	64620	75466	$\frac{1}{19}$
389	16164 ( $\infty$ )	38435 (140977)	45902	15924	14028	24976	41328	64620	3292	1467	1029	833	64620	75855	$\frac{1}{19}$
390	16287 ( $\infty$ )	38585 (142302)	46703	16355	14028	24976	41328	64620	3365	1481	1036	838	64620	76245	$\frac{1}{19}$
391	16413 ( $\infty$ )	38725 (143649)	47529	16807	14028	24976	41328	64620	3441	1496	1043	842	64620	76636	$\frac{1}{19}$
392	16539 ( $\infty$ )	38875 (145018)	48381	17282	14028	24976	41328	64620	3520	1511	1050	847	64620	77028	$\frac{1}{19}$
393	16665 ( $\infty$ )	39025 (146409)	49259	17782	14028	24976	41328	64620	3602	1526	1057	852	64620	77421	$\frac{1}{19}$
394	16794 ( $NaN$ )	39175 (147823)	50166	18309	14028	24976	41328	64620	3689	1541	1064	856	64620	77815	$\frac{1}{19}$
395	16926 ( $\infty$ )	39315 (149261)	51103	18866	14028	24976	41328	64620	3778	1556	1072	861	64620	78210	$\frac{1}{19}$
396	17055 ( $\infty$ )	39465 (150723)	52071	19454	14028	24976	41328	64620	3872	1572	1079	866	64620	78606	$\frac{1}{19}$
397	17190 ( $\infty$ )	39615 (152209)	53072	20077	14028	24976	41328	64620	3970	1588	1087	871	64620	79003	$\frac{1}{19}$
398	17322 ( $\infty$ )	39765 (153721)	54107	20738	14028	24976	41328	64620	4073	1604	1094	875	64620	79401	$\frac{1}{19}$
399	17460 ( $\infty$ )	39915 ( $NaN$ )	55180	21440	14028	24976	41328	64620	4180	1621	1102	880	64620	79800	$\frac{1}{19}$
400	17595 ( $NaN$ )	40065 (156823)	56290	22187	14028	24976	41328	64620	4293	1637	1109	885	64620	80200	$\frac{1}{19}$

## Appendix B

**Full version of Theorem 1** (Barg and Yu, 2013, Theorem 3.1). The upper bound of  $s^{\alpha,\beta}(r)$  is given by the solution of the semidefinite programming problem:

$$\begin{aligned}
& \text{maximize } 1 + \frac{1}{3}(s_1 + s_2) \\
& \text{subject to} \\
& 3 + G_i^r(\alpha)s_1 + G_i^r(\beta)s_2 \geq 0, \quad i = 1, 2, \dots, p
\end{aligned} \tag{32}$$

$$s_j \geq 0, \quad j = 1, \dots, 6 \tag{33}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (s_1 + s_2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (s_3 + s_4 + s_5 + s_6) \succeq 0 \tag{34}$$

$$\begin{aligned}
& S_i^r(1, 1, 1) + S_i^r(\alpha, \alpha, 1)s_1 + S_i^r(\beta, \beta, 1)s_2 + S_i^r(\alpha, \alpha, \alpha)s_3 \\
& + S_i^r(\alpha, \alpha, \beta)s_4 + S_i^r(\alpha, \beta, \beta)s_5 + S_i^r(\beta, \beta, \beta)s_6 \succeq 0, \quad i = 0, 1, \dots, p
\end{aligned} \tag{35}$$

where

- $s_1, \dots, s_6$  are real variables;
- the notation  $\succeq 0$  means the matrix on the left side of  $\succeq$  is semidefinite positive;

- $p$  is a positive integer and  $p$  is chosen to be 5 in practice (Barg and Yu, 2013, 2014);
- $G_i^r$  is the so-called *Gegenbauer polynomial* in the univariate polynomial ring  $\mathbb{Q}[t]$  ( $\mathbb{Q}$  denotes the field of rational numbers), which can be defined recursively:

$$G_0^r(t) = 1, G_1^r(t) = t, \text{ and } G_i^r(t) = \frac{(2i + r - 4)tG_{i-1}^r + (i - 1)G_i^r}{i + r - 3} \quad (i \geq 2);$$

- $S_i^r(u, v, t)$  is a  $(p - i + 1) \times (p - i + 1)$  matrix defined by

$$S_i^r(u, v, t) = \frac{1}{6} \sum_{\sigma} Y_i^r(\sigma(u, v, t))$$

where again

- $\sigma$  is all permutations of 3 elements;
- $Y_i^r(u, v, t)$  is a  $(p - i + 1) \times (p - i + 1)$  matrix and the  $(k, j)$ -element of  $Y_i^r(u, v, t)$  is defined by

$$u^k v^j ((1 - u^2)(1 - v^2))^{\frac{i}{2}} G_i^{r-1} \left( \frac{t - uv}{\sqrt{(1 - u^2)(1 - v^2)}} \right).$$

#### History of the SDP formulation (32–35).

- The constraints (32–33) formulate a linear programming problem. This linear programming method is the widely-used Delsarte’s method in coding theory (Delsarte et al., 1977).
- The linear programming method (32–33) was used to compute the upper bound of spherical two-distance sets in (Musin, 2009).
- The matrices  $S_i^r$  in (35) were developed in (Bachoc and Vallentin, 2008) and they were used to formulate an SDP problem for solving the *kissing number* problem.
- The SDP formulation (32–35) was used in (Barg and Yu, 2013) to solve the upper bounds of spherical two-distance sets. And in (Barg and Yu, 2014), the authors applied the same formulation to solve the upper bounds of equiangular sets.